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A non-perturbative approach to non-commutative scalar field theory

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Abstract

Non-commutative Euclidean scalar field theory is shown to have an eigenvalue sector which is dominated by a well-defined eigenvalue density, and can be described by a matrix model. This is established using regularizations of $\mathbb{R}^{2n}_\theta$ via fuzzy spaces for the free and weakly coupled case, and extends naturally to the non-perturbative domain. It allows to study the renormalization of the effective potential using matrix model techniques, and is closely related to UV/IR mixing. In particular we find a phase transition for the ϕ^4 model at strong coupling, to a phase which is identified with the striped or matrix phase. The method is expected to be applicable in 4 dimensions, where a critical line is found which terminates at a non-trivial point, with nonzero critical coupling. This provides evidence for a non-trivial fixed-point for the 4-dimensional NC ϕ^4 model.

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Contents

1 Introduction

The idea of considering quantum field theory on "quantized" or non-commutative spaces (NCFT) was put forward a long time ago [1], and has been pursued vigorously in the past years; see e.g. [2, 3] for a review. A very intriguing phenomenon which was found in this context is the so-called UV/IR mixing [5], linking the usual UV divergences to new singularities in the IR. On a technical level, it arises because of a very different behavior of planar and non-planar diagrams, which must be distinguished on NC spaces. The planar diagrams are essentially the same as in the commutative case. The non-planar diagrams

however lead to oscillating integrals, which are typically finite as long as the external momentum p is non-zero, but become divergent in the limit $p \to 0$. This leads to serious obstacles to perturbative renormalization [5]. Furthermore, it appears to signal an additional phase denoted as "striped phase" [6–8], which arises as the minimum of the effective action is no longer at zero momentum.

Because UV/IR mixing is so generic in the NC case, it is necessary to come to terms with it and to find suitably adapted quantization methods. The first step is clearly a suitable regularization of the models. This can be achieved by parametrizing the fields in terms of finite matrices, which is very natural on NC spaces. Several such methods are available by now, using e.g. using fuzzy spaces, non-commutative tori, etc. The action for scalar fields is then a functional of a hermitean matrix ϕ, where the potential $TrV(\phi)$ looks like a "pure" matrix model with $U(N)$ invariance, which is however broken by the kinetic term. The UV/IR mixing is expected to be recovered in the continuum limit.

Such a regularization has been used recently to confirm numerically the non-trivial phase structure mentioned above in the non-commutative ϕ^4 model [9, 10]. There has also been remarkable progress on the analytical side using matrix techniques: A modified ϕ^4 model with an explicit IR regulator term in the action was shown to be perturbatively renormalizable [11], and certain self-dual models of NC field theory were solved exactly using a matrix model formulation [13]. For gauge theories, the applicability of well-known techniques from random matrix theory has also been shown in simple cases [14, 15]. For the ϕ^4 model, a similar approach using random matrix theory does not seem possible at first sight, lacking $U(N)$ invariance. Nevertheless, it was conjectured in [9] that the striped phase should be identified with a "matrix phase" for the fuzzy sphere, where the action appears to be dominated by a pure potential model in that phase. Hence a simple analytical approach which allows to study also scalar NCFT's with non-trivial phase structure and UV/IR mixing is highly desirable. In particular, it seems that the obvious parallels between NCFT and pure matrix models due to UV/IR mixing have not yet been fully exploited, apart from integrable cases [13].

The aim of this paper is to show that there is indeed a simple matrix model description which captures a certain sector of scalar NCFT, due to UV/IR mixing. This suggests a new approach to scalar NCFT which not only provides new insights, but also new tools to study the renormalization of the effective potential. The starting point is an appropriate parametrization for the fields: since ϕ is a hermitian matrix, it can be diagonalized as $\phi = U^{-1} diag(\phi_i) U$ where ϕ_i are the real eigenvalues. Hence the field theory can be reformulated in terms of the eigenvalues ϕ_i and the unitary matrix U. The main observation of this paper is now the following: the probability measure induced on the (suitably rescaled) eigenvalues ϕ_i from the path integral is sharply localized, and described by an ordinary, simple matrix model. This means that only fields ϕ with a particular eigenvalue distribution characterized by a certain function $\rho(s) : [-1, 1] \to \mathbb{R}^+$ contribute significantly to the (euclidean) path integral. While this is plausible using the above parametrization, it is a nontrivial statement which is only true in the con-commutative regime. It is established first for the free case (which therefore *does* know about non-commutativity, contrary to a common belief) and extends immediately to the interacting case at least on a perturbative level. This is directly related to UV/IR mixing, since non-planar contributions to the eigenvalue observables are suppressed by the oscillatory factors, while the planar contributions can be described by a simple matrix model without kinetic term. It is quite obvious that this will extend also to

the non-perturbative level, in a suitable domain. This suggests that scalar NCFT can be characterized by a single function $\rho(s)$.

We then work out some simple applications of this approach, which do not require long computations. In the weak coupling regime, this leads to a very simple method of computing the mass renormalization using matrix model techniques. In particular the standard one-loop result for the mass renormalization in the ϕ^4 model is recovered in a non-standard way, and finds a natural interpretation in the matrix model. Unfortunately the running of the coupling does not seem to admit such a simple computation. In any case, we will argue that there exists a scaling limit with a non-trivial correlation length in the continuum limit, suggesting the existence of a renormalized ϕ^4 model in 2 and 4 dimensions.

Extending these results to the non-perturbative regime, we find a phase transition for the ϕ^4 model in 2 and 4 dimensions, to a phase which is tentatively identified with the striped or matrix phase of [6, 9]. This can be compared with numerical results available for the fuzzy sphere in 2 dimension, with reasonable agreement which confirms the overall picture. The method is expected to work better in 4 dimensions, due the stronger divergences which are crucial for our derivation. In the 4-dimensional case, the critical line is found to terminate at a non-trivial point, with nonzero critical coupling $g_c \neq 0$. This is expected to be a sound prediction, suggesting the existence of a non-trivial fixed-point for the 4-dimensional NC ϕ^4 model, and hence renormalizability in an appropriate sense.

To summarize, while the dominance of planar diagrams in NCFT is very well known, the description of the eigenvalue sector in terms of a simple matrix model (39) appears to be new and is very practical. This provides an alternative approach to some of the results of [6–8] on the phase structure in NCFT, confirming the rough picture of a phase transition towards a phase which breaks translational invariance. However, the phase transition is predicted to be higher-order, as opposed to [6–8]. We find in addition a critical coupling $g_c \neq 0$ in 4 dimensions, and it would be very interesting to verify this numerically.

This paper is organized as follows. Section 2 provides some background recalling the UV/IR mixing, and introduces the matrix regularizations used later. Section 3 is the core of this paper: after identifying the suitable observables, we show that the eigenvalue distribution of free fields is given by Wigner's semi-circle law, which allows to replace the kinetic term by the matrix model (34). Interactions are included in Section 3.2. The corresponding reformulation of scalar NCFT using eigenvalue and angle coordinates is discussed in general in Section 3.3 and 3.4, including an intuitive semi-classical picture. This is then applied to the ϕ^4 model in section 4, and related to some standard results for hermitean matrix models. These allow in Section 5 to obtain the mass renormalization in a very simple way. The phase transition is studied in detail for 2 and 4 dimensions in section 6, and compared with numerical results for the fuzzy sphere. We conclude in Section 7 with further remarks and an outlook.

2 NC scalar fields and UV/IR mixing

Consider scalar field theory on the non-commutative Moyal plane $\mathbb{R}^d_\theta$ in even dimensions, with action

$$S = \int d^d x \left(\frac{1}{2} \partial_i \phi \partial_i \phi + \frac{1}{2} m^2 \phi^2 + \frac{g}{4} \phi \star \phi \star \phi \star \phi \right) . \tag{1}$$

Here ϕ is a function on $\mathbb{R}^d$, and the $\star-$ product is the standard Moyal product of functions on $\mathbb{R}^d$, which can be written as

$$(a \star b)(x) = \int \frac{d^4k}{(2\pi)^4} \int d^4y\, a(x+\tfrac{1}{2}\theta\cdot k)\, b(x+y)\, \mathrm{e}^{\mathrm{i}k\cdot y}\ , \tag{2}$$
$$(\theta\cdot k)_i = \theta_{ij}k_j\ , \quad k\cdot y = k_i y_i\ , \quad \theta_{ij} = -\theta_{ji}\ .$$

This can be understood as a pull-back of the operator product of 2 operators a and b from a representation of the underlying (Heisenberg) algebra

$$[x_i, x_j] = i\theta_{ij}, \tag{3}$$

using a suitable quantization map. We assume that θ_{ij} is non-degenerate in this paper.

The model (1) written down above is not well-defined as it stands, and needs regularization. The simplest way to proceed is to use a sharp UV cut-off Λ, which leads to standard computations and will be justified below. The perturbative quantization of (1) differs from the commutative case by the fact that planar and non-planar diagrams must be distinguished. The reason is that commuting 2 plane waves with wavenumbers k and k' produces a factor $e^{-ik\theta k'}$, which makes non-planar loops convergent *for generic external momenta*. More explicitly, the basic one-loop planar and non-planar self-energy diagrams (without counting symmetry factors) are [5]

$$\Gamma_P^{(2)} := \int \frac{d^dk}{(2\pi)^d}\, \frac{1}{k^2+m^2},$$
$$\Gamma_{NP}^{(2)}(p) := \int \frac{d^dk}{(2\pi)^d}\, \frac{e^{ik\theta p}}{k^2+m^2}. \tag{4}$$

$\Gamma_{NP}^{(2)}(p)$ is finite as long as $p \neq 0$ due to the oscillating term, but has an IR singularity as $p \to 0$ because the k-integral is then divergent as usual. This is known as UV/IR mixing [5], and appears to be a central feature of NC field theories. It is a serious obstacle to perturbative renormalization, which was only overcome recently in a modified ϕ^4 model [11].

In this paper, we shall try to turn this UV/IR mixing into a virtue, and point out that it is closely related to an interesting property of the scalar field ϕ in the operator formulation, which seems very useful and does not hold for ordinary field theories: The dynamical field ϕ has a well-defined eigenvalue distribution upon quantization, which is governed by a simple matrix model and can be studied using a saddle-point analysis.

2.1 Matrix regularization of $\mathbb{R}^n_\theta$

Recall that if the non-commutative algebra (3) is represented on a (infinite-dimensional) Hilbert space $\mathcal{H}$, the integral is given by the suitably normalized trace:

$$\int d^dx f(x) = (2\pi)^{d/2}\sqrt{\det\theta}\; Trf = (2\pi\theta)^{d/2}\; Trf \tag{5}$$

where f in the rhs is the operator version of $f(x)$ (as obtained e.g. using the Weyl quantization map). This is a manifestation of the Bohr-Sommerfeld quantization condition,

relating the volume of the phase space to the dimension of the Hilbert space. The last line holds for $Sp(d)$- invariant θ_{ij}, which we assume in this paper for simplicity.

We want to approximate this using some *finite-dimensional* matrix algebra. This can be achieved e.g. using a suitable scaling limit of the fuzzy sphere for $d = 2$, or more generally fuzzy $\mathbb{C}P^n$ for $d = 2n$, see Appendix A. Indeed fuzzy $\mathbb{C}P^n \to \mathbb{R}^{2n}_\theta$ in a suitable limit, where θ_{ij} turns out to be invariant under $Sp(2n)$, and the propagator is the usual one with a sharp momentum cutoff. Another possible regularization is using so-called NC lattices which are products of certain ("fuzzy") NC tori [18], see also Appendix A. These are technically somewhat easier to handle, but lead to a modified behavior of the propagators for large momenta; then the most general θ_{ij} can be obtained. In all these regularizations, ϕ is a hermitian $\mathcal{N} \times \mathcal{N}$ matrix in some finite matrix algebra $Mat(\mathcal{N}, \mathbb{C})$, and the model (1) is replaced by a matrix model (where the kinetic term breaks the $U(\mathcal{N})$ symmetry). In particular, the trace is now over the $\mathcal{N}$ -dimensional Hilbert-space $\mathcal{H} = \mathbb{C}^{\mathcal{N}}$, where $\mathcal{N}$ is related to the cutoff Λ and θ in a specific way (154), (118) depending on the regularization. Then $V := \int d^dx 1 = (2\pi\theta)^{d/2}\, \mathcal{N}$, and we can write

$$\frac{1}{V}\int d^dx f(x) = \frac{1}{\mathcal{N}}\, Trf. \tag{6}$$

In particular, integrals of the type $\int \phi^{2n}$ depend only on the eigenvalues of ϕ, and these are the observables we want to study.

To make the paper most readable, the regularization using fuzzy $\mathbb{C}P^n$ with sharp momentum cutoff Λ will be understood, while using the conventional language of $\mathbb{R}^d_\theta$ as much as possible. The results for general (non-degenerate) θ_{ij} and somewhat modified propagators would be qualitatively the same.

3 The eigenvalue distribution of the scalar field

The basic idea is the following: Having regularized the model (1) in terms of a finite-dimensional hermitean matrix ϕ, we can diagonalize it as

$$\phi = U^{-1}(\phi_i)U \tag{7}$$

where U is a unitary $\mathcal{N} \times \mathcal{N}$ matrix, and $(\phi_i) \equiv diag(\phi_1, ..., \phi_{\mathcal{N}})$ is diagonal with real eigenvalues. The integration measure in the path integral can now be written as $\int \mathcal{D}\phi e^{-S} = \int d\phi_i \Delta^2(\phi_i) \int dU e^{-S}$, where $\Delta^2(\phi_i) = \prod_{i<j}(\phi_i - \phi_j)^2$ is the Vandermonde-determinant and dU the Haar measure for $SU(\mathcal{N})$. We are interested in the probability measure or effective action for these eigenvalues, induced by this path integral. For this purpose, consider e.g. the expectation values

$$\langle \int d^dx \phi^{2n}(x)\rangle = \frac{1}{Z}\int \mathcal{D}\phi \exp(-S)(\int d^dx \phi^{2n}(x)). \tag{8}$$

They are strongly divergent normally but make sense in the regularized (matrix) case; in particular, we do *not* want to replace the $\phi^{2n}(x)$ by some renormalized objects but simply keep track of their dependence on the cutoff. Since they depend only on the eigenvalues of the field ϕ in the matrix representation, we can determine the effective eigenvalue distribution by studying such observables. This turns out to be non-trivial already in the free case:

3.1 The free case

We compute the observables (8) for $g=0$ with a sharp UV - cutoff Λ using Wicks theorem. This involves in general planar and non-planar diagrams. The simplest case is[1]

$$
\begin{aligned}
\langle\int d^dx\phi^2(x)\rangle &= V\int_0^\Lambda \frac{d^dp}{(2\pi)^d}\,\frac{1}{p^2+m^2} = \frac{V}{2^{d-1}\pi^{d/2}(d/2-1)!}\int_0^\Lambda dp\frac{p^{d-1}}{p^2+m^2} \\
&=: c(m,\Lambda)V\,\Lambda^{d-2}. \qquad (9)
\end{aligned}
$$

This formal result will be fully justified in Appendix A using regularizations of $\mathbb{R}_\theta^{d=2n}$ in terms of fuzzy $\mathbb{C}P^n$ (153) or fuzzy tori. Here V denotes the regularized volume of $\mathbb{R}_\theta^d$, and $c=c(m,\Lambda)$ is of order 1 for $d\geq 3$, and $c=O(\ln\Lambda)$ for $d=2$. More precisely, in 4 dimensions we have

$$
\langle\int d^4\phi^2(x)\rangle = \frac{Vm^2}{8\pi^2}\int_0^{\Lambda/m} du\frac{u^3}{u^2+1} = \frac{V\Lambda^2}{16\pi^2}\left(1-\frac{m^2}{\Lambda^2}\ln(1+(\frac{\Lambda}{m})^2)\right) \qquad (10)
$$

and in 2 dimensions

$$
\langle\int d^2\phi^2(x)\rangle = \frac{V}{2\pi}\int_0^{\Lambda/m} du\frac{u}{u^2+1} = \frac{V}{4\pi}\ln(1+\frac{\Lambda^2}{m^2}) \qquad (11)
$$

(the subleading behavior is modified for the regularization using NC tori, see Appendix A). Therefore

$$
c(m,\Lambda) = \left\{\begin{array}{ll} \frac{1}{16\pi^2}\left(1-\frac{m^2}{\Lambda^2}\ln(1+\frac{\Lambda^2}{m^2})\right), & d=4 \\ \frac{1}{4\pi}\ln(1+\frac{\Lambda^2}{m^2})), & d=2. \end{array}\right. \qquad (12)
$$

Next, consider

$$
\begin{aligned}
\langle\int d^dx\phi(x)^4\rangle &= \langle\int d^dx\phi(x)^4\rangle_{Planar} + \langle\int d^dx\phi(x)^4\rangle_{Non-Planar} \\
&= 2V\int_0^\Lambda \frac{d^dp}{(2\pi)^d}\,\frac{1}{p^2+m^2}\,\Gamma_P^{(2)}(p) + V\int_0^\Lambda \frac{d^dp}{(2\pi)^d}\,\frac{1}{p^2+m^2}\,\Gamma_{NP}^{(2)}(p),
\end{aligned}
$$

which is obtained by summing over all complete contractions of a vertex with 4 legs. There are 2 planar and one non-planar such contractions, the latter being given by joining the external legs of the non-planar self-energy diagram (4). The planar contribution is simply

$$
\langle\int d^dx\phi(x)^4\rangle_{Planar} = 2V\;c^2\Lambda^{2(d-2)}, \qquad (13)
$$

since $\Gamma_P^{(2)}(p)$ is independent of p. On the other hand, the non-planar contribution is subleading (this will be discussed in detail below), since $\Gamma_{NP}^{(2)}(p)$ is finite except for the singularity at $p\to 0$. This is clearly related to UV/IR mixing, even though we are considering only the free case up to now. We therefore expect

$$
\frac{\langle\frac{1}{V}\int d^dx\phi(x)^4\rangle}{\langle\frac{1}{V}\int d^dx\phi(x)^2\rangle^2} = 2\quad + O(\frac{1}{c\Lambda^{d-2}}). \qquad (14)
$$

[1] we basically assume that θ_{ij} is non-degenerate in this paper, which implies that d is even. However, some of the results extend to the degenerate case, which will be pursued elsewhere

More generally, consider

$$\frac{\langle \frac{1}{V}\int d^dx\phi(x)^{2n}\rangle}{\langle \frac{1}{V}\int d^dx\phi(x)^2\rangle^n} = \frac{\langle \frac{1}{V}\int d^dx\phi(x)^{2n}\rangle_{Planar}}{\langle \frac{1}{V}\int d^dx\phi(x)^2\rangle^n} + \frac{\langle \frac{1}{V}\int d^dx\phi(x)^{2n}\rangle_{Non-Planar}}{\langle \frac{1}{V}\int d^dx\phi(x)^2\rangle^n}.$$

The first term is of order one, and simply counts the number $N_{Planar}(2n)$ of planar contractions of a vertex with $2n$ legs. The non-planar contributions always involve oscillatory integrals, and do not contribute to the above ratio in the large Λ limit. Let us discuss them in more detail: Assume that the cut-off is much larger than the NC scale,

$$\Lambda^2\theta \gg 1, \tag{15}$$

which will be understood from now on. By rescaling the momenta $k' = k/\Lambda$, the above ratios for $d \geq 3$ have the form

$$R_{NP} := \frac{\langle \frac{1}{V}\int d^dx\phi(x)^{2n}\rangle_{Non-Planar}}{\langle \frac{1}{V}\int d^dx\phi(x)^2\rangle^n} \approx \int_0^1 \frac{d^dk'_1}{(k'_1)^2}...\frac{d^dk'_n}{(k'_n)^2}e^{i\Lambda^2\sum k'_i\Theta k'_j} \quad \to 0 \;\text{ as }\; \Lambda \to \quad \infty, \tag{16}$$

which vanish for large Λ due to the rapidly oscillating exponential. The integrals are over the unit ball resp. hypercube. This is established more carefully in Appendix B, where we show that $R_{NP} = O(1/\Lambda)$ (at least) for $d \geq 3$. In 2 dimensions, these considerations are more delicate as the above ratio R_{NP} vanishes only logarithmically, and divergences may also arise in the IR. One must then assume furthermore that e.g. $\frac{m}{m_\theta}$ is fixed while $\Lambda \to \infty$, where

$$m_\theta^2 := \frac{1}{\theta} \tag{17}$$

is the non-commutative mass scale. Due to this weaker logarithmic behavior, the fluctuations are expected to be larger for $d = 2$ than for $d = 4$. We refrain here from more precise estimates which should be made in the context of the regularized models, see Appendix B.

With all these assumptions, we conclude that

$$\frac{\langle \frac{1}{V}\int d^dx\phi(x)^{2n}\rangle}{\langle \frac{1}{V}\int d^dx\phi(x)^2\rangle^n} = \frac{\langle \frac{1}{V}\int d^dx\phi(x)^{2n}\rangle_{Planar}}{\langle \frac{1}{V}\int d^dx\phi(x)^2\rangle^n} = N_{Planar}(2n) \tag{18}$$

in the large Λ limit. Notice that this result is very different from the conventional field theory: the above calculation with a naive cutoff would have the same contributions from planar and non-planar diagrams. Then the total number of contractions of e.g. $\langle\phi^{2n}\rangle$ is of order $2^n n! \gg N_{Planar}(2n)$, which would invalidate the conclusions below.

Next, consider expectation values of products $\langle(\int d^dx\phi(x)^{2n_1})...(\int d^dx\phi(x)^{2n_k})\rangle$. We claim that this factorizes in the large Λ limit,

$$\begin{aligned}\frac{\langle\big(\frac{1}{V}\int d^dx\phi(x)^{2n_1}\big)...\big(\frac{1}{V}\int d^dx\phi(x)^{2n_k}\big)\rangle}{\langle \frac{1}{V}\int d^dx\phi(x)^2\rangle^{n_1}...\langle \frac{1}{V}\int d^dx\phi(x)^2\rangle^{n_k}} &= \frac{\langle \frac{1}{V}\int d^dx\phi(x)^{2n_1}\rangle}{\langle \frac{1}{V}\int d^dx\phi(x)^2\rangle^{n_1}}\cdots\frac{\langle \frac{1}{V}\int d^dx\phi(x)^{2n_k}\rangle}{\langle \frac{1}{V}\int d^dx\phi(x)^2\rangle^{n_k}} \\ &= N_{Planar}(2n_1)\;...\;N_{Planar}(2n_k)\end{aligned} \tag{19}$$

This can be seen again in terms of contractions, because propagators joining different vertices must satisfy additional momentum constraints as opposed to the disjoint contractions, therefore only the disjoint contributions survive in the large Λ limit. This amounts essentially to the cluster property.

Now recall that in the non-commutative case, the integral is given by the suitably normalized trace (6). Hence the integrals above depend only on the eigenvalues of ϕ, and we obtain statements about the induced eigenvalue distribution. The above observables (19) can be written in the form

$$(\frac{1}{\mathcal{N}}Tr\phi^{n_1})...(\frac{1}{\mathcal{N}}Tr\phi^{n_k}), \tag{20}$$

and completely determine the effective probability measure for the eigenvalues of ϕ. We can certainly write any such expectation value as

$$\langle(\frac{1}{\mathcal{N}}Tr\phi^{n_1})...(\frac{1}{\mathcal{N}}Tr\phi^{n_k})\rangle = \int d\phi_1...d\phi_n\mu(\phi_1,...,\phi_n)\ (\frac{1}{\mathcal{N}}\sum_i \phi_i^{n_1})...(\frac{1}{\mathcal{N}}\sum_j \phi_j^{n_k}) \tag{21}$$

for some measure $\mu(\phi_1,...,\phi_n)$, which we would like to determine for large Λ. In order to absorb the infinities we introduce a scaling constant

$$\alpha_0^2(m) = 4c\Lambda^{d-2} = \begin{cases} \frac{1}{4\pi^2}\,\Lambda^2\left(1-\frac{m^2}{\Lambda^2}\ln(1+\frac{\Lambda^2}{m^2})\right), & d=4 \\ \frac{1}{\pi}\ln(1+\frac{\Lambda^2}{m^2}), & d=2 \end{cases} \tag{22}$$

and write[2]

$$\phi = \alpha_0\varphi. \tag{23}$$

Then (9) gives

$$\langle\frac{1}{\mathcal{N}}Tr\varphi^2\rangle = \frac{1}{4}. \tag{24}$$

Now all expectation values $\langle\frac{1}{\mathcal{N}}Tr\varphi^{2n}\rangle$ are finite and have a well-defined limit $\mathcal{N}\to\infty$. We can then describe the eigenvalues of φ (in increasing order) by an eigenvalue distribution,

$$\varphi(s) = \varphi_j, \qquad s = \frac{j}{\mathcal{N}}, \qquad s\in[0,1]. \tag{25}$$

Then e.g.

$$\frac{1}{\mathcal{N}}\mathrm{Tr}f(\varphi) = \frac{1}{\mathcal{N}}\sum_i f(\varphi_i) \to \int_0^1 ds\ f(\varphi(s)) \tag{26}$$

in the large $\mathcal{N}$ limit. The measure μ now becomes a measure $\mu[\varphi(s)]$ on the space of (increasing) functions $\varphi(s) : [0,1] \mapsto \mathbb{R}$. To find this measure $\mu[\varphi(s)]$, we first note that the factorization property (19) implies that the measure μ is *localized*, i.e.

$$\langle f(\varphi_i)\rangle = f(\overline{\varphi}(s)) \tag{27}$$

for any function f, where $\overline{\varphi}(s)$ is the (sharp and dominant) saddle-point or maximum of $\mu[\varphi]$. This saddle-point $\overline{\varphi}(s)$ corresponds to a density of eigenvalues

$$\rho(\overline{\varphi}) = \frac{ds}{d\overline{\varphi}}, \qquad \int_{-\infty}^{\infty}\rho(p)dp = 1. \tag{28}$$

[2] α_0 is *not* the wavefunction renormalization

The expectation value of the above observables is then given by

$$\langle \frac{1}{\mathcal{N}} \mathrm{Tr} f(\varphi) \rangle = \int dp \rho(p)\, f(p), \tag{29}$$

or $\langle \frac{1}{\mathcal{N}} \mathrm{Tr}\varphi^n \rangle = \int_0^1 ds\, \overline{\varphi}^n$; for example, $\langle \int ds \varphi(s)^2 \rangle = \int ds \overline{\varphi}(s)^2 = \frac{1}{4}$. We want to determine this saddle-point $\overline{\varphi}(s)$. This can be extracted from the above results: (18) implies that

$$\langle \int_0^1 ds \varphi(s)^{2n} \rangle = \int \rho(p) p^{2n} dp = (\frac{1}{4})^n\; N_{Planar}(2n). \tag{30}$$

There is a unique eigenvalue distribution with these properties, given by the famous Wigner semi-circle law

$$\rho(p) = \begin{cases} \frac{2}{\pi}\sqrt{1-p^2} & p^2 < 1 \\ 0, & \text{otherwise.} \end{cases} \tag{31}$$

This means that the eigenvalues of ϕ are distributed correspondingly in the interval

$$\phi_i \in [-\alpha_0, \alpha_0]. \tag{32}$$

It is fun to verify (30) explicitly for small n; indeed

$$\int_{-1}^1 \rho(x) x^{2n} dx = \frac{\Gamma(\frac{1}{2}+n)}{\Gamma(2+n)} = \frac{1}{4^n}\; N_{Planar}(2n). \tag{33}$$

In general, this follows also from the basic properties of matrix models: Consider the Gaussian matrix model with action

$$\boxed{\tilde{S}_0 = f_0(m) + \tfrac{2\mathcal{N}}{\alpha_0^2} Tr\phi^2} \tag{34}$$

where $\phi = \alpha_0 \varphi$ is a $\mathcal{N}\times\mathcal{N}$ matrix. Here $f_0(m)$ is some numerical function of m which will be determined below, and α_0 depends on m via $c = c(m)$. $\tilde{S}_0$ will find another interpretation in Section 3.3. $\tilde{S}_0$ is known to reproduce precisely the eigenvalue distribution (31) in the large $\mathcal{N}$ limit [19], and (30) follows because again only planar diagrams contribute. Indeed, one finds again e.g.

$$\langle \frac{1}{\mathcal{N}} Tr\phi^2 \rangle = \alpha_0^2 \langle \frac{1}{\mathcal{N}} Tr\varphi^2 \rangle = \frac{1}{4}\, \alpha_0^2 \tag{35}$$

etc.; we refer to the vast literature available on this subject, e.g. [19–21].

We conclude that if one is only interested in the eigenvalues, the free action $S_0 = \int d^dx \frac{1}{2}(\partial_i\phi\partial_i\phi + m^2\phi^2)$ can be replaced by the effective action $\tilde{S}_0$ in (34). Moreover, one can determine $f_0(m)$ such that the partition function is also recovered as

$$Z = \int \mathcal{D}\phi e^{-S_0} = \int \mathcal{D}\phi e^{-\tilde{S}_0} = \int \mathcal{D}\phi \exp(-f_0(m) - \frac{2\mathcal{N}}{\alpha_0^2} Tr\phi^2); \tag{36}$$

this will be understood better in Section 3.3. To summarize, we noted that the observables (19) depend only on the eigenvalues; the factorization property implies that the dominating contribution comes from a well-defined eigenvalue distribution, which is via (30) identified as the Wigner-distribution corresponding to a Gaussian matrix model. We can then write

down an effective Gaussian matrix model which reproduces all the expectation values for these observables. Note that the details of the propagator enter only through α_0, and the basic result depends only on the degree of divergence.

It is interesting to compare this with the commutative case, where the eigenvalues are replaced by the values of the field $\phi(x)$ at each point, i.e. N^d variables for a lattice regularization. Using again a cutoff Λ, we would have

$$\frac{\langle \frac{1}{V}\int d^dx\phi(x)^{2n}\rangle}{\langle \frac{1}{V}\int d^dx\phi(x)^2\rangle^n} = N_{Planar}(2n) + N_{Nonplanar}(2n) = \int_{-\infty}^{\infty} e^{-s^2/2}s^{2n} \approx 2^n n! \gg N_{Planar}(2n).$$

This could be reproduced by a Gaussian distribution $\tilde{S}_0^C = \frac{1}{2\alpha_0^2}\int d^dx\phi(x)^2$. The crucial difference is that this would describe N^d independent variables (which is false of course, but ok for these observables) with a very flat distribution, while in the NC case we consider only $\mathcal{N} = N^{d/2}$ eigenvalues which are collective and intrinsically non-local variables, governed by an effective action with a sharp potential[3]; note also the explicit factor $\mathcal{N}$ in (34). This will become more explicit in the following sections, which do not apply to the commutative case.

Let us try to interpret this result. It may be surprising that already the free scalar NCFT is apparently very different from the commutative case, even though they are supposed to be the same from the star-product point of view. The reason is that we are looking at statistical properties of the operator representation of the wavefunctions, which is related to their point-wise values only for low momenta where the functions are "almost-commutative". Therefore the non-classical properties of the high-energy modes are responsible for this property. This already indicates UV/IR mixing: the modes with $k \approx 0$ are suppressed upon quantization because they tend to have the "wrong" eigenvalue distribution (in particular $\phi_{k=0} \propto \mathbb{1}$ has only one eigenvalue).

The existence of a well-defined eigenvalue distribution and its matrix model description will generalize easily to the interacting case at least on a perturbative level, and is very plausible also in the non-perturbative domain. Furthermore it suggests new and practical insights to the coupling constants and their renormalization. This will be discussed next.

3.2 Interactions

If we include interactions of the form

$$S_{int}(\phi) = \frac{g_n}{n}\int d^dx\phi^n(x) = \frac{g_n}{n}(2\pi\theta)^{d/2}Tr\phi^n,$$

the results (34), (36) for the free case generalize at least on a perturbative level: consider

$$Z_{int} = \int \mathcal{D}\phi e^{-(S_0+S_{int})} = \int \mathcal{D}\phi e^{-S_0}\left(1 - \frac{g_n}{n}(2\pi\theta)^{d/2}Tr\phi^n + ...\right) \tag{37}$$

Hence the expansion of $e^{-S_{int}}$ in powers of g_n leads to additional terms of the form (19) or (20), which are again evaluated by the rule (29) for the measure ρ of the free case[4]. We

[3] recall that the saddle-point approximation for the matrix models is good for the eigenvalues, but not for the matrix elements

[4] This is valid as long as the coupling constants g_n are small enough so that the eigenvalue distribution remains unchanged. This should be enough to determine the perturbation series, where g_n can be arbitrarily small. In particular, it suffices to determine mass renormalizations etc. as seen below.

can write a similar formula to evaluate observables such as $\langle \frac{1}{\mathcal{N}} Tr\phi^{2n}\rangle_g$ in the presence of g_n. Therefore the results of Section 3.1 apply, and one can simply replace S_0 by $\tilde{S}_0$ in (37). We conclude that at least perturbatively, the eigenvalue sector of the NCFT with action

$$S = \int d^dx \frac{1}{2}(\partial_i\phi\partial_i\phi + m^2\phi^2) + S_{int} \tag{38}$$

for any polynomial interaction is described by the *effective matrix model*

$$\boxed{\tilde{S}(\phi) = f_0(m) + \tfrac{2\mathcal{N}}{\alpha_0^2(m)} Tr\phi^2 + S_{int}(\phi)} \tag{39}$$

where $\alpha_0 = \alpha_0(m)$ is given by (22), in the large $\mathcal{N}$ limit. This is expected to be correct as long as the eigenvalue distribution is close enough to the free one (31). The partition function is similarly given by

$$Z_{int} = \int \mathcal{D}\phi \exp(-f_0(m) - \frac{2\mathcal{N}}{\alpha_0^2(m)} Tr\phi^2 + S_{int}(\phi)). \tag{40}$$

This defines by the usual matrix model technology an analytic function in the couplings g_n. In the later sections, this will allow us in particular to determine some renormalization properties of the potential is a very simple way. We will also explore non-perturbative implications such as phase-transitions, hoping that this will give at least qualitatively correct results.

Scaling and relevant couplings. Let us try to estimate the impact of the interaction terms to the eigenvalue distribution. In the matrix-regularizations of NCFT used below, we will have $\mathcal{N} \sim N^{d/2}$ and $\Lambda = O(\sqrt{N})$. Therefore the following scaling behavior holds

$$\phi \sim \alpha_0 \sim \Lambda^{\frac{d}{2}-1}, \quad \mathcal{N} = \Lambda^d \tag{41}$$

using (23). We assume this scaling also in the interacting case in the weakly-coupled domain, and check for which g_n this is self-consistent. Then (39) essentially becomes $\tilde{S} \sim Tr(\mathcal{N}\varphi^2 + g_n\alpha_0^n\varphi^n) =: Tr(\mathcal{N}V(\varphi))$, omitting constants of order 1 (including θ, which is *not* assumed to scale). Here φ is of order one. The resulting eigenvalue distribution is governed by $V(\varphi)$ [19], and it will remain near the Gaussian fixed point resp. Wigner's law provided the bare couplings satisfy $g_n \leq \mathcal{N}/\alpha_0^n \sim \Lambda^\delta$, where

$$\delta = d + n - \frac{nd}{2} \tag{42}$$

is just engineering dimension of g_n. As usual, this means that relevant or marginal couplings with $\delta \geq 0$ are "safe" and expected to be renormalizable, while irrelevant couplings with $\delta < 0$ must be fine-tuned and are expected to be non-renormalizable.

In general, it will be safe to use (39) as long as the shape of the resulting eigenvalue distribution $\overline{\varphi}(s)$ is close to the Wigner law. For a qualitatively different shape, one should expect corrections to (39).

In particular, it is quite clear that turning on some small coupling g_4 must be compensated by a suitable "mass renormalization" in order to preserve the shape of $\overline{\varphi}(s)$. This allows to determine the mass renormalization, which will be explored in Section 5. But before discussing these issues, let us look at the above results from a different perspective:

3.3 Angle-eigenvalue coordinates in field space

One of the merits of the NC field theory is that it naturally suggests new coordinates in field space, which are very different from the usual "local" fields $\phi(x)$ in the commutative case. This is particularly obvious using a regularization in terms of finite-dimensional matrices $\phi \in Mat(\mathcal{N}, \mathbb{C})$. Then there is a natural action of the $SU(\mathcal{N})$ group $\phi \to U^{-1}\phi U$, which can be seen as NC version of the symplectomorphisms. Even though this is not a symmetry due to the kinetic term, it suggests to parametrize ϕ in terms of eigenvalues and "angles", which are very non-local coordinates. This change of variables leads very naturally to the picture we found above. In particular, the non-trivial measure factor in the path integral due to the Jacobian makes the existence of a non-trivial eigenvalue distribution very plausible.

We start from the simple fact that any hermitian matrix ϕ can be diagonalized,

$$\phi = U^{-1}(\phi_i)U \tag{43}$$

where $U \in SU(\mathcal{N})$ and (ϕ_i) is a diagonal matrix, with eigenvalues ϕ_i. We can moreover assume that the eigenvalues of (ϕ_i) are ordered; then the matrix U is unique up to phase factors $K \cong U(1)^{\mathcal{N}-1}$, provided the eigenvalues ϕ_i are non-degenerate. This leads to the following definition of the orbits

$$\mathcal{O}(\phi) := \{U^{-1}(\phi_i)U; \;\; U \in SU(\mathcal{N})\} \cong SU(\mathcal{N})/K \tag{44}$$

where K is the stabilizer group of (ϕ_i). These are compact homogeneous spaces. Then the partition function can be written as

$$\begin{aligned} Z &= \int \mathcal{D}\phi \; \exp(-S(\phi))) = \int d\phi_i \Delta^2(\phi_i) \int dU \exp(-S(U^{-1}(\phi_i)U)) \\ &= \int d\phi_i dU \exp(\sum_{i\neq j} \log|\phi_i - \phi_j| - S(U^{-1}(\phi_i)U)) \\ &= \int d\phi_i \tilde{F}(\phi_i) \exp(-(2\pi\theta)^{d/2} \sum_i V(\phi_i) + \sum_{i\neq j} \log|\phi_i - \phi_j|), \end{aligned} \tag{45}$$

where dU is the integral over $\mathcal{N} \times \mathcal{N}$ unitary matrices; similar manipulations were also done in [9]. We introduced here the function

$$\tilde{F}(\phi) := \int dU \exp(-S_{kin}(U^{-1}(\phi)U)) =: e^{-\tilde{\mathcal{F}}(\phi)}, \tag{46}$$

which by definition depends only on the eigenvalues of ϕ. The last form is justified because $\tilde{F}(\phi)$ is positive. It satisfies

$$\tilde{\mathcal{F}}(\phi + c) = \tilde{\mathcal{F}}(\phi), \quad \tilde{\mathcal{F}}(-\phi) = \tilde{\mathcal{F}}(\phi). \tag{47}$$

Moreover, $\tilde{\mathcal{F}}(\phi_i)$ is *analytic* in the ϕ_i because the space is compact, and invariant under exchange of the eigenvalues. We can therefore expect that it approaches some nice classical functional $\tilde{\mathcal{F}}[\phi(s)]$ in the large $\mathcal{N}$ limit, where $\phi(s)$ is the function in *one* variable which is related to ϕ_i as in (25).

We can now read off the induced action for the eigenvalues,

$$\tilde{S}(\phi_i) = \tilde{\mathcal{F}}(\phi_i) - \sum_{i \neq j} \log|\phi_i - \phi_j| + (2\pi\theta)^{d/2} \sum_i V(\phi_i) \tag{48}$$

The log - term in (48) could also be absorbed by defining

$$\mathcal{F}(\phi_i) = \tilde{\mathcal{F}}(\phi_i) - \sum_{i \neq j} \log|\phi_i - \phi_j|. \tag{49}$$

In particular, note that the log-term in (48) strongly suppresses degenerate eigenvalues, and this cannot be compensated by any analytic $\tilde{\mathcal{F}}(\phi_i)$. Therefore the saddle-points of $\tilde{S}(\phi_i)$ corresponds to some non-degenerate eigenvalue distribution. Furthermore the kinetic term (encoded in $\tilde{\mathcal{F}}(\phi_i)$) strongly suppresses jumps in this eigenvalue distribution[5], therefore we expect it to approach some smooth function $\overline{\phi}(s)$ in the limit $\mathcal{N} \to \infty$. Furthermore, we expect this eigenvalue distribution to have compact support after a suitable rescaling:

$$\overline{\phi}(s) = \alpha\overline{\varphi}(s) \tag{50}$$

where α denotes the maximal eigenvalue. This typically happens for matrix models, and appears to be true also in this context as shown below. Indeed since $\tilde{\mathcal{F}}(\phi)$ only depends on the eigenvalues, one can trivially interpret it as a function of any hermitean matrix $\phi = U^{-1}(\phi_i)U$, and rewrite the partition function as an *ordinary* matrix model with formal $U(\mathcal{N})$ symmetry,

$$Z = \int d\phi_i \exp(-\tilde{S}(\phi_i)) = \int \mathcal{D}\phi \exp(-\tilde{\mathcal{F}}(\phi) - (2\pi\theta)^{d/2}\, TrV(\phi)). \tag{51}$$

The last step is of course completely formal, and we are only allowed to determine observables depending on the eigenvalues with this action. These are determined by the "effective action" $\tilde{S}(\phi_i)$, since the degrees of freedom related to U are integrated out. This strongly suggests that much of the information about the quantum field theory, in particular the phase transitions and the thermodynamic properties, are determined by $\tilde{S}(\phi_i)$ and the resulting eigenvalue distribution in the large $\mathcal{N}$ limit. Note that this is essentially a one-dimensional problem, governed by the (unknown) functional $\mathcal{F}[\phi(s)]$ and V. The advantage of this formulation is that it is very well suited to include interactions, and naturally extends to the non-perturbative domain.

Now assume we know $\mathcal{F}(\phi_i)$; one can then look for the saddle-points of $\tilde{S}(\phi_i)$, determined by

$$\frac{\delta\tilde{\mathcal{F}}}{\delta\phi_i} + (2\pi\theta)^{d/2}\, V'(\phi_i) = \sum_{j \neq i} \frac{1}{\phi_j - \phi_i} \tag{52}$$

and ask if they are localized enough to dominate the observables. If the existence of a sharp eigenvalue distribution is established, the full path integral in (45) would be dominated by the integral over the corresponding $SU(\mathcal{N})$ orbit $\mathcal{O}(\phi_i)$. This in turn should allow to recover

[5] At strong coupling however, we will find a phase transition to a distributions with one gap, corresponding to some "striped" phase

also other properties such as correlation functions, by integrating over the corresponding $\mathcal{O}(\phi_i)$ which is compact. This will be discussed further in Section 3.4.

We can now relate this to the results of Section 3.1: All observables of the eigenvalues as considered there are determined by the "effective action" $\tilde{S}(\phi_i)$ above. The result of Sections 3.1 and 3.2, in particular the factorization property (19), says that there is indeed a well-defined eigenvalue distribution in the non-commutative domain[6], i.e. as long as $\Lambda^2\theta \gg 1$. Comparing with the effective action for the free case (34), we find

$$\tilde{\mathcal{F}}(\phi) = f_0(m) + \frac{2\mathcal{N}}{\alpha_0^2} Tr\phi^2 \tag{53}$$

which certainly reproduces all admissible observables for the potential $V(\phi) = \int \frac{1}{2}m^2\phi^2$.

This formula may appear strange, since $\tilde{\mathcal{F}}(\phi)$ should of course be independent of V. The reason is that this relation (53) has been established only "on-shell", for eigenvalue distributions close to the Wigner law for the free case. It is not clear how well this works for large deviations from that case. Later we will use this form also for eigenvalue distributions which are quite different from the free one, where one should expect corrections to (53). The appropriate way to use (53) for some given eigenvalue distribution is therefore to determine m such that the corresponding free distribution matches best the one under consideration. It would be extremely interesting to know more about the functional $\tilde{\mathcal{F}}(\phi)$.

The dominance of a given orbit $\mathcal{O}(\phi_i)$ in (45) is clearly related to UV/IR mixing: naively, the action has a minimum at $\phi = const\ 1\!\!1$; however if the volume-factors from the path integral are taken into account (which happens at one loop), this zero-momentum state is actually highly suppressed, and the dominating field configurations have nontrivial position-dependence (i.e. momentum), due to the nontrivial eigenvalue distribution. Note that this argument is completely non-perturbative. Furthermore, depending on the form of the potential $V(\phi)$ the dominating eigenvalue distribution may be connected or consist of disjoint pieces. These would clearly correspond to different phases. This picture will be made more quantitative below, and we will be able to identify the "striped" phase of [6].

3.3.1 Interpretation of the orbits $\mathcal{O}(\phi_i)$.

Consider the reduced model (46) for a given orbit $\mathcal{O}(\phi_i)$ with fixed eigenvalue distribution, whose (free) energy is given by $\tilde{\mathcal{F}}(\phi)$. Intuitively, we can interpret this model as follows: consider a classical fluid $\phi(x)$ on a compact space (due to the regularization e.g. on $\mathbb{C}P^n$ or some torus) with prescribed "density"

$$\rho(p) = \frac{1}{V} \int d^dx\ \delta(\phi(x) - p), \tag{54}$$

corresponding to the eigenvalue distribution. Then $\rho(p)$ is essentially the density of eigenvalues (28), at least in the semi-classical limit. Note in particular that the action of the classical volume-preserving diffeomorphisms on $\phi(x)$ can be approximated by $SU(\mathcal{N})$, since any configuration with given ρ can be obtained using $SU(\mathcal{N})$. Entropy will favor mixing, but the "kinetic energy" suppresses mixing. In a small region of space, the global constraint (54) is quite irrelevant, and the fluid will behave like a fluid with the same action

[6]This is not true for conventional field theory with $\theta = 0$, even though the formulation of this section is still possible for e.g. the commutative limit of fuzzy spaces.

but without the constraint. However, if we fix the field on a large part of the volume, this must be compensated in the remaining space. This is clearly an IR effect, suppressing very large wavelengths. Therefore we expect that the theory behaves like an "ordinary" field theory in small enough regions of space and may hence describe ordinary local physics, however it is certainly different globally. This is quite interesting and encouraging for possible applications in elementary particle physics.

In order to go beyond the computations in the following sections, one should therefore study the reduced models (46) in more detail, and see to what extent they approximate a scalar field theory. Note that quite generally if the scale α is increased, the configurations with short wavelength will be more strongly suppressed, leading a long correlation length; on the other hand for small α, the correlation length will be short. Therefore there should indeed exist some suitable scaling $\alpha(\mathcal{N})$ which gives a macroscopic correlation length in the limit $\mathcal{N} \to \infty$. The relation with the intuitive picture presented in Section 3.3 and in particular (54) can be seen best using coherent states or projectors (see e.g. [25, 26, 28]), in particular for the regularizations with $\mathbb{C}P^n_N$. It would be very interesting to combine these methods with the approach in the present paper.

The "ground state" of (46) on a given orbit $\mathcal{O}(\phi_i)$ is some very smooth function with the given density, which solves the e.o.m.

$$[\Delta\phi, \phi] = 0. \tag{55}$$

This has many interesting solutions: One class is given by solutions of the free wave equation $\Delta\phi = c\phi$. In particular, the (non-commutative) spherical harmonics with suitable eigenvalue density are solutions also on the above orbits $\mathcal{O}(\phi_i)$. However there are other solutions, for example any solution of $\Delta\phi = f(\phi)$ for arbitrary f solves (55); in particular any diagonal matrix does (in the usual basis). A careful study of these issues should lead to a better understanding of (46), and hence to improvements of the simple results presented below.

3.4 Relating the matrix model to physical observables

Before analyzing further the matrix models (39), we should try to relate them to the physically interesting quantities such as mass and coupling constants. Recall that the definition of mass on a non-commutative space is not obvious, since the Lorentz-invariance is generally broken. However on Euclidean $\mathbb{R}^{2n}_\theta$ with $Sp(2n)$- invariant θ_{ij} one can define a *correlation length* in terms of correlation functions for 2 suitably localized Gaussian wave-packets[7] $\phi(x)$ of size $1/\sqrt{\theta}$, say, or e.g. coherent states for the fuzzy spaces $\mathbb{C}P^n_N$:

$$\langle\phi(x_1)\ \phi(x_2)\rangle = \int d\phi_i \Delta^2(\phi_i) \int_{\mathcal{O}(\phi_i)} dU e^{-S(U^{-1}(\phi_i)U)} \langle\phi(x_1), U^{-1}(\phi_i)U\rangle\langle\phi(x_2), U^{-1}(\phi_i)U\rangle. \tag{56}$$

Here $\langle\phi(x), U^{-1}(\phi_i)U\rangle \propto Tr(\phi(x)U^{-1}(\phi_i)U)$ denotes the inner product on the NC space. Note again that only the kinetic part of the action is non-trivial here, and we will assume

[7] These "test-functions" can be moved transitively on the NC spaces using the translational symmetry which is unbroken. On $\mathbb{R}^{2n}_\theta$ with $U(n)$- invariant θ_{ij} resp. their fuzzy versions $\mathbb{C}P^n_N$, the residual unbroken $U(n)$ rotational symmetry is maximal and large enough to ensure that the correlation length is independent of the direction.

in the following that the full path integral is dominated by some orbit $\mathcal{O}(\phi_i)$. The mass can then be identified as the inverse correlation length, provided it is $\ll m_\theta$. Similarly, one could define the coupling constants in terms of correlation functions of e.g. 4 such wave-packets. In general, it is plausible that the interesting low-energy observables depend only on the eigenvalue distribution, and can be determined in principle by an integral over the orbit $\mathcal{O}(\phi_i)$.

The question of renormalizability is then roughly whether one can scale the (finitely many) couplings g_n with the cutoff $\Lambda \to \infty$ in such a way that the correlation length and all the other observables at low energy (or at the scale of non-commutativity set by θ) approach a well-defined limit. To answer this question of course requires control over all these correlation functions. In this paper, we try to proceed as much as possible without resorting to perturbation theory. In view of the above results on the eigenvalue distribution, it seems very plausible that the scaling of the bare couplings m and g_n must be determined such that *the shape of the dominating eigenvalue distribution, i.e. the normalized function* $\overline{\varphi}(s)$ *(50) is fixed as* $\Lambda \to \infty$. These scalings should be easily accessible with matrix model techniques if we know the matrix model, in particular $\tilde{\mathcal{F}}(\phi)$. Moreover, it seems plausible that this should even be sufficient to guarantee "renormalizability" in the above sense, provided the field theory can indeed be reduced to the orbit $\mathcal{O}(\phi_i)$ as discussed in Section 3.3.

To make explicit computations, we have to use (39) resp. (53) for the time being, i.e. we have to require that $\overline{\varphi}(s)$ respectively its related eigenvalue density (28) is close enough to the Wigner law (31). Then the "physical" mass m_R resp. correlation length can be identified without computing such correlation functions: it should be given by the bare mass of the corresponding free theory *with the same maximal eigenvalue* $\alpha_0(m_R)$, for the same cutoff Λ. That is, m_R is determined by

$$\alpha = \alpha_0(m_R) \tag{57}$$

where α (50) will depend on the couplings of the interacting matrix model. This will be elaborated in Section 4. We will apply this prescription (57) even if the function $\overline{\varphi}(s)$ is not close to the free one in this paper, hoping that it is mainly the "size" of the eigenvalue distribution which determines the correlation length. This is plausible in view of the classical picture discussed in Section 3.3.1.

We want to point out again the following consequence of this picture: (56) is strongly suppressed for zero momentum $\phi_0 \propto \mathbb{1}$, since $\langle \phi_0, U^{-1}(\phi_i)U\rangle = 0$ for the dominating eigenvalue-distribution. On the other hand for non-zero momentum, the eigenvalues are non-degenerate, and the above inner product is non-zero. This suppression of zero momentum is clearly related to UV/IR mixing, and suggests that indeed only localized wave-packets should be used.

4 Example: the ϕ^4 model

Consider now the model

$$S = \int d^d x \left(\frac{1}{2}\partial_i\phi\partial_i\phi + \frac{1}{2}m^2\phi^2 + \frac{g}{4}\phi^4\right)$$

$$= \int d^dx \left(\frac{1}{2}\partial_i\phi\partial_i\phi + \frac{1}{2}m_R^2\phi^2 + \left(\frac{1}{2}\Delta m^2\phi^2 + \frac{g}{4}\phi^4\right)\right) \tag{58}$$

where we have introduced a "renormalized" mass $m_R^2 = m^2 - \Delta m^2$. Its eigenvalue sector according to the above results is described by

$$\tilde{S} = f_0(m_R) + \frac{2\mathcal{N}}{\alpha_0(m_R)^2}\, Tr\phi^2 + (2\pi\theta)^{d/2} Tr\left(\frac{1}{2}\Delta m^2\phi^2 + \frac{g}{4}\phi^4\right). \tag{59}$$

While m_R is arbitrary in principle, we will adjust it in order to minimize the expected errors due to our only partial knowledge of $\mathcal{F}(\phi_i)$. Note that in the regularization using fuzzy $\mathbb{C}P^n$, we will find (154)

$$\Lambda = \sqrt{\frac{2N}{\theta}} \tag{60}$$

where N is related to $\mathcal{N}$ via (145), and a similar result $\Lambda = \sqrt{\frac{\pi N}{\theta}}$ using NC tori (118) where $\mathcal{N} = N^{d/2}$. Therefore

$$\frac{\mathcal{N}}{\alpha_0^2} = O(N\theta^{(d-2)/2}) = O(\Lambda^2\theta^{d/2}) \tag{61}$$

up to log- corrections.

To solve this model, we first rescale ϕ as

$$\phi = \alpha_g\varphi \tag{62}$$

such that the saddle-point solution for φ will have an eigenvalue distribution with range[8] $[-1, 1]$. Rewriting the matrix model (59) as

$$\tilde{S} = f_0(m_R) + \mathcal{N}Tr\left(\frac{m'^2}{2}\varphi^2 + \frac{g'}{4}Tr\varphi^4\right). \tag{63}$$

we have

$$\begin{aligned}
\alpha_g^2\left(\frac{2\mathcal{N}}{\alpha_0(m_R)^2} + (2\pi\theta)^{d/2}\frac{1}{2}\Delta m^2\right) &= \frac{\mathcal{N}}{2}m'^2, &(64)\\
(2\pi\theta)^{d/2}g\alpha_g^4 &= \mathcal{N}g', &(65)\\
m^2 &= m_R^2 + \Delta m^2. &(66)
\end{aligned}$$

The model (63) is well-known and can be solved with standard methods from random matrix theory, see e.g. [19,21]. It is again governed by a single saddle-point in the eigenvalue sector, with eigenvalue distribution $\overline{\varphi}_g(s)$ resp. $\rho_g(p)$. Note that now $m'^2 < 0$ is allowed. Assuming that $g' > 0$, there are 2 cases corresponding to distinct phases of the model which we are discussed below.

It is interesting to note that even small $g' < 0$ is admissible for this matrix model as long as $m'^2 > 0$. This indicates analyticity in g', so that perturbation theory does make sense in the weakly coupled phase.

[8]in the disordered phase discussed at present, see Section 4.1

4.1 Phase 1 ("single-cut"): $m'^2 > 0$, or $m'^2 < 0$ with $\frac{m'^4}{4g'} < 1$

In this case the eigenvalue density is given by [19]

$$\rho_g(\varphi) = \frac{1}{2\pi}(g'\varphi^2 + \frac{g'}{2} + m'^2)\sqrt{1^2 - \varphi^2}. \tag{67}$$

We have imposed that the range of φ be $[-1, 1]$ as explained above, which using standard formulas [19] implies

$$1 = \frac{2}{3g'}(-m'^2 + \sqrt{m'^4 + 12g'}) \tag{68}$$

i.e.

$$g' = \frac{4}{3}(4 - m'^2) \tag{69}$$

Note that the matrix model (63) has 2 independent parameters, which can be chosen either as m'^2 and g', or e.g. α_g and g'. We have chosen to work with α_g, therefore m'^2 and g' are not independent. In particular, note that $\frac{m'^4}{4g'} < 1$ for $g' < 16$ due to (69), therefore this phase will be the "weakly-coupled" phase in Section 5.

Inserting $m'^2 = 4 - \frac{3}{4}\, g'$ in (64) and using (65) gives

$$\frac{2\alpha_g^2}{\alpha_0^2} + (2\pi\theta)^{d/2}\alpha_g^2\frac{1}{2\mathcal{N}}\Delta m^2 = 2 - \frac{3}{8}\,(2\pi\theta)^{d/2}\frac{\alpha_g^4}{\mathcal{N}}\,g. \tag{70}$$

So far, m_R was arbitrary, and it shouldn't matter for fixed "bare parameters" m and g. However, the relation (39) is expected to be good only if the eigenvalue distribution is close to the free one corresponding to m_R. We should therefore choose m_R accordingly, and a good choice is

$$\alpha_0(m_R) = \alpha_g \tag{71}$$

as in (57). This guarantees that the eigenvalue distribution of the interacting model (63) has the same range as the free one obtained from m_R, hence it is "close". Then (70) simplifies further, and using $\Delta m^2 = m^2 - m_R^2$ we have

$$\begin{aligned}(2\pi\theta)^{d/2} g\, \alpha_0^4(m_R) &= \mathcal{N} g' &(72)\\ m^2 + \frac{3}{4} g\, \alpha_0^2(m_R) &= m_R^2. &(73)\end{aligned}$$

These are 2 equations in 4 unknowns (the function $\alpha_0(m_R)$ being known (22)), and we can basically choose any 2 of them to parametrize the model.

4.2 Phase 2 ("2 cuts"): $m'^2 < 0$ with $\frac{m'^4}{4g'} > 1$

At $4g' = m'^4$ with $m'^2 < 0$ the eigenvalue density breaks up into 2 disjoint peaks, which are concentrated around the 2 minima. For $4g' < m'^4$ these peaks have finite distance, and moreover spontaneous symmetry breaking of $Tr\phi$ occurs with non-zero "occupation" in both peaks [22]. This clearly describes a different phase of the model. The semi-classical interpretation in the picture of Section 3.3.1 would be a 2-fluid model, with a large surface energy at the contact surface due to the kinetic term. It is very plausible that this will

develop some kind of (generalized) striped pattern upon mixing, and we conjecture this to be the "striped" phase of [6], or equivalently the "matrix phase" of [9]. Note that the assumption in [9] of 2 delta-like peaks is qualitatively very close, but strictly ruled out by the log-terms in (48).

It is even possible to consider negative g' in this model [19], reflecting the analyticity in g'. If g' becomes too negative there is another phase transition, which will not be considered any further here.

5 Weak coupling and renormalization

We first consider the weak coupling regime, and try to get some insights into the renormalization of the ϕ^4 model. In particular, one would like to understand better the relation of the above approach with the usual concepts of running coupling constants etc. The natural scaling parameter here is the size of the matrices $\mathcal{N} \sim N^{d/2}$, which is related to the UV-cutoff Λ through (60) for the regularization using fuzzy $\mathbb{C}P^n$. We can therefore ask how the bare parameters m^2, g etc. must be scaled with Λ such that the "low-energy physics" remains invariant, and we could define the corresponding beta-functions.

In the weak coupling regime, the shape of the eigenvalue distribution $\overline{\varphi}$ (50) resp. ρ_g (67) should certainly be kept close to the free one, given by the Wigner law. As discussed in Section 3.4, the correlation length resp. "renormalized mass" m_R can then be obtained simply by matching the size of the eigenvalue spectrum α_g with the one for the free case with the same cutoff,

$$\alpha_g = \alpha_0(m_R) \tag{74}$$

see (57). Then m_R measures the correlation length of the free model which best fits the eigenvalue distribution. This should be very reasonable, since we are looking at a weak coupling expansion where g' can be taken arbitrarily small.

The corresponding running of the original parameters m and g depends on the dimension, and will be worked out below. For the mass, this is very easy in our approach. Unfortunately, it is not obvious how to relate the coupling g resp. g' to e.g. 4-point functions or scattering processes; this could be supplemented perturbatively by a conventional RG computation, cp. [11, 12]. Also, note that there is no wavefunction-renormalization in this approach, as there are only 2 coupling constants in the matrix model (63). On the other hand the computations below are much simpler than any standard field-theoretic computations, and provide non-trivial information on the model without having to resort to perturbation theory.

5.1 2 dimensions

To find the scaling of m^2 and g we can use (72) and (73), where $\alpha_g = \alpha_0(m_R)$ is already built in. The running of the mass can be read off from (73),

$$m^2 = m_R^2 - \frac{3}{4} g\alpha_0^2(m_R) = m_R^2 - \frac{3}{4\pi} g \ln(1 + \frac{\Lambda^2}{m_R^2}) \tag{75}$$

Recall that m_R is the "physical", renormalized mass which is supposed to be finite while $\Lambda \to \infty$. We see explicitly the expected logarithmic divergence of the bare mass. Note that

this is an all-order result in g, despite appearance.

It is interesting to compare this with a conventional one-loop calculation. Noting that our interaction term is $\int \frac{g}{4}\phi^4$ (which is more natural in the matrix model context) rather than $\int \frac{g}{4!}\phi^4$, one would find

$$\delta m_{P+NP}^2 = 12\,\frac{g}{4}\int \frac{d^2k}{(2\pi)^2}\frac{1}{k^2+m_R^2} = \frac{3}{4\pi} g\ln(1+\frac{\Lambda^2}{m_R^2}) \tag{76}$$

from the 8 planar plus 4 non-planar contractions. It is interesting to see that this one-loop computation agrees precisely with our procedure of matching the eigenvalue distribution with the free one, even though we used the non-perturbative matrix model results (67) ff. One might have expected that only the planar diagrams contribute to the mass renormalization; however since the "mass" is obtained as the $p \to 0$ limit of the 1PI 2-point functions (for fixed Λ), the planar and non-planar diagrams coincide and both contribute.

Note that if we use another regularization such as NC tori, we should use the modified propagator (128) rather than the sharp cutoff in (11). This would lead to a somewhat modified formula for $\alpha_0(m_R)$, however the corresponding mass renormalization (75) would still coincide with the one-loop result since the same $\alpha_0(m_R)$ enters in both calculations.

Next consider the bare coupling g. Since the matrix model provides no simple relation between g and the 4-point function, the latter would have to be obtained e.g. by a perturbative calculation as usual. However since g is not expected to run in 2 dimensions, we simply interpret g as "physical" coupling constant, fixing the scale by m_θ. The relation with the matrix model coupling g' is given by (72), which for large Λ and using (60) is

$$g \;=\; \frac{\Lambda^2\pi}{16(\ln(\frac{\Lambda}{m_R}))^2}\; g'. \tag{77}$$

In particular if we keep g finite, we see that $g' \to 0$ very rapidly as $\Lambda \to \infty$. This means that the eigenvalue density ρ_g (67) is very close to the free one given by Wigner's law, and we expect that the above relations are reliable in this weak-coupling phase of ϕ^4 in 2 dimensions[9].

5.2 4 dimensions

Using the same procedure, (73) gives now the expected quadratic divergence of the bare mass

$$m^2 = m_R^2 - \frac{3}{16\pi^2}\,\Lambda^2\left(1-\;\frac{m_R^2}{\Lambda^2}\;\ln(\frac{\Lambda^2}{m_R^2})\right)\; g. \tag{78}$$

This is again an all-order result, and agrees again perfectly with a conventional one-loop computation (see e.g. (3.11) of [5] after replacing $g \to g/6$). However the bare coupling g is now also expected to run, being logarithmically divergent at one loop. Recall also that we cannot see any wavefunction-renormalization as there are only 2 parameters in the matrix model. The relation between g and the matrix-parameter g' is now given by

$$g' = \frac{2}{\pi^2}\left(1-2\,\frac{m_R^2}{\Lambda^2}\ln(\frac{\Lambda}{m_R})\right)^2\; g \approx \frac{2}{\pi^2}\,g \tag{79}$$

[9]Recall however that the non-planar diagrams are suppressed only logarithmically (14) in 2D, which is much weaker than in 4D

for $\Lambda \gg m_R$, where (60) and $\mathcal{N} = N^2/2$ (145) has been used. This means that (keeping in mind some possibly logarithmic renormalization of g) our procedure to derive (78) is justified in a perturbative sense, since we can make g' arbitrarily small so that again ρ_g is very close to Wigner's law.

It is quite remarkable that according to the above results, the mass is renormalized only at first order in the bare coupling g, even though the derivation is non-perturbative. While this is expected for $d = 2$, this seems very surprising for $d = 4$; it would indicate that NC ϕ^4 in 4 dimensions is much better behaved than in the commutative case. This can be traced back to our procedure of determining the mass m_R through $\alpha = \alpha_0(m_R)$. While an exact calculation of the correlation length on a given orbit $\mathcal{O}(\phi)$ would certainly modify this result, the basic picture seems clear and simple. Unfortunately this analysis does not give us the relation between g and the "physical" coupling as obtained by the 4-point function, which probably requires further renormalization.

It is tempting here to conjecture that fixing g' and m' or equivalently the eigenvalue distribution $\overline{\varphi}$ in a suitable way suffices to define a non-trivial NC field theory in 4 dimensions. We will find further evidence for this in Section 6.1.

We conclude that while this approach certainly needs further work and thus far provides only a partial window into NCFT, it suggests a very simple and compelling approach to renormalization in the NC case. This may help to overcome the difficulties at higher order found in [5].

5.3 Higher dimensions

It is also illuminating to try to generalize the above considerations to dimensions higher than 4, where the commutative ϕ^4 model is no longer renormalizable. The relations (66) and (72) are still valid, and give

$$m^2 = m_R^2 - g\, O(\Lambda^{d-2}) \tag{80}$$

and

$$g' = O(\Lambda^{d-4})\, g. \tag{81}$$

Clearly our matrix model (63) makes sense only for $g' = O(1)$, which would require $g = O(\Lambda^{4-d})$. This is just the canonical dimension of g. If we again assume that g' and hence the shape of the eigenvalue distribution should be independent of Λ, we see that g must be fine-tuned to at least $O(\Lambda^{4-d})$. This is in accord with the expected non-renormalizability.

6 The phase transition.

Now consider the phase transition of the matrix model (63), which is known [21,23] to have a phase transition between the 1-cut and the 2-cut phase at

$$m'^4 = 4g'. \tag{82}$$

This is expected to be the phase transition between the disordered and striped phase of Gubser and Sondhi. Combining with (69) this gives

$$g' = \frac{4}{3}(4 - m'^2) = \frac{1}{4}m'^4 \tag{83}$$

which has 2 solutions for m'^2; only the negative one

$$m'^2 = -8 \tag{84}$$

is relevant here and marks the phase transition. The corresponding coupling is[10]

$$g' = 16. \tag{85}$$

Note that there is still a free parameter in the model, which we can take to be m_R resp. α. Plugging this in (72) gives

$$(2\pi\theta)^{d/2} g\alpha_0^4(m_R) = 16\mathcal{N}, \tag{86}$$

$$m^2 + \frac{3}{4}\, g\, \alpha_0^2(m_R) = m_R^2, \tag{87}$$

and solving the first equation for α_0 we obtain

$$m^2 + 3\sqrt{\frac{\mathcal{N}g}{(2\pi\theta)^{d/2}}} = m_R^2(g), \tag{88}$$

where m_R^2 is a function of g via (86). This defines the critical line. The corresponding phase transition is third order [23] in the variables g' and m'. This is in contrast with [6–8], who argued for a first-order phase transition.

Note that $m'^2 < 0$ means that we are quite far from the perturbative domain, and the eigenvalue distribution is significantly changed from Wigner's law (31). In this regime the replacement (34) for the kinetic term has not been tested, and we cannot expect the results to be exact. Furthermore, we expect the "bare" parameters m, g to have some non-trivial scaling in $\mathcal{N} \sim N^{d/2}$ on the critical line corresponding to the RG flow, which is not obvious. In any case, it is quite reasonable that this description is at least qualitatively correct. To proceed, we have to discuss the dimensions separately, starting with the most interesting case of 4 dimensions.

6.1 4 dimensions

We consider first the regularization using fuzzy $\mathbb{C}P^2$ discussed in Appendix A, which corresponds to a sharp cutoff. In this case we have $\mathcal{N} = N^2/2$, and $\alpha_0^2 = O(\Lambda^2)$ in $d = 4$ (22). By looking at the equations (86), (87) and

$$\Lambda^2 = \frac{2N}{\theta} \tag{89}$$

which holds for this regularization of $\mathbb{R}^4_\theta$, it is quite obvious that there should be solutions which scale as

$$m^2 \propto \Lambda^2 \propto N, \quad g \quad \text{fixed.} \tag{90}$$

Hence we expect a phase transition at finite coupling g and the standard quadratic running of the mass. To find a closed equation for the critical line, we have to use (22) for $\alpha_0 = \alpha_0(m_R)$,

$$\frac{4\pi^2\alpha_0^2(m_R)}{\Lambda^2} = 1 - \frac{m_R^2}{\Lambda^2}\ln(1 + \frac{\Lambda^2}{m_R^2}) \tag{91}$$

[10] Recall that $g' < 16$ is the weakly coupled single-cut phase as pointed out in section 4.1.

This cannot be solved explicitly for m_R, but we can use (88) for m_R^2 which together with $\mathcal{N} = N^2/2$ and (89) gives

$$\frac{m^2}{\Lambda^2} + \frac{3}{4\pi}\sqrt{\frac{g}{2}} = \frac{m_R^2}{\Lambda^2}. \tag{92}$$

Since $m_R^2 \geq 0$, this makes sense only for

$$\frac{m^2}{\Lambda^2} > -\frac{3}{4\pi}\sqrt{\frac{g}{2}}. \tag{93}$$

Plugging this in (91) and using (86)

$$\alpha_0^2(m_R) = \frac{N}{\pi\theta}\sqrt{\frac{2}{g}} = \frac{\Lambda^2}{\pi\sqrt{2g}} \tag{94}$$

we get

$$2\pi\sqrt{\frac{2}{g}} = 1 - \left(\frac{m^2}{\Lambda^2} + \frac{3}{4\pi}\sqrt{\frac{g}{2}}\right)\ln\left(1 + \left(\frac{m^2}{\Lambda^2} + \frac{3}{4\pi}\sqrt{\frac{g}{2}}\right)^{-1}\right). \tag{95}$$

This is indeed consistent with the scaling (90). Since the rhs is $\in (0,1]$, this has a solution only[11] for $2\pi\sqrt{\frac{2}{g}} \leq 1$, i.e.

$$g \geq g_c = 8\pi^2. \tag{96}$$

In terms of the dimensionless parameter $\tilde{m}^2 = m^2/\Lambda^2$, the corresponding critical mass is $\tilde{m}_c^2 = -\frac{3}{2}$. Expanding $g = g_c + \delta g,\ \ \tilde{m}^2 = \tilde{m}_c^2 + \delta\tilde{m}^2$ the critical line (95) is given by

$$\delta g = -\frac{32\pi^2}{3}\,\delta\tilde{m}^2 + ... \tag{97}$$

for small variations. This is plotted in Figure 1.

Let us try to assess the validity of this result. The basic arguments underlying this result are expected to be good in 4 dimensions as long as $m_R^2 \ll \Lambda^2$ (unlike in 2 dimensions, see below and Appendix B). This is satisfied here since as $g \to g_c$ from above, (95) implies that

$$\frac{m_R^2}{\Lambda^2} = \frac{m^2}{\Lambda^2} + \frac{3}{4\pi}\sqrt{\frac{g}{2}} \quad \to \quad 0 \tag{98}$$

using (92), and in particular $m_R = 0$ at $g = g_c$. Therefore the replacement of the free action with the matrix model (34) is essentially justified, apart from the modified eigenvalue distribution $\overline{\varphi}(s)$. It would be very interesting to estimate the effects of this modified $\overline{\varphi}(s)$ more rigorously, e.g. by estimating the integrals over the compact orbits $\mathcal{O}(\phi_i)$. The existence of $g_c \neq 0$ can also be understood simply by noting that the critical line is characterized by a specific eigenvalue distribution which is different from Wigner's law, which holds for $g = 0$. Therefore the critical line[12] cannot end at $g = 0$. While the existence of a critical point should be a sound prediction, the relation between m^2 and g

[11] the rhs of (95) is simply $c(\Lambda, m_R)$ as defined in (9), which will very generally satisfy such bounds.

[12] note that $m^2 \propto \Lambda^2$ here, as in the weakly-coupled regime. This is different in 2 dimensions, where the critical line is in a different scaling regime from the weakly-coupled case, and no such conclusion can be drawn.

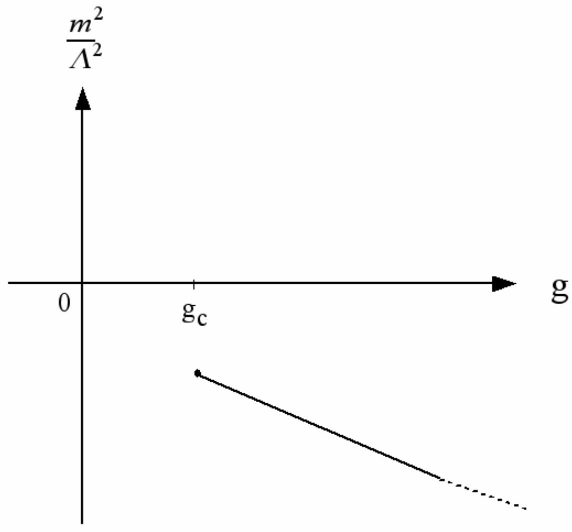

Figure 1: The critical line in 4D with critical point

on the critical line cannot be expected to be precise. If we would use the propagator for the lattice regularization (128), the details of (95) would change but the results would be qualitatively the same. In particular the critical coupling could be evaluated using (132).

The existence of a critical point terminating the critical line at $g_c \neq 0$ is certainly intriguing. Since the critical line will be stable under the RG flow, its endpoint should correspond to a non-trivial RG fixed point for the non-commutative ϕ^4 theory, and the correlation length is expected to diverge[13]. That limiting model should be non-trivial, since the eigenvalue distribution $\overline{\varphi}(s)$ is different from the free case. It is furthermore plausible following the discussion in [6] that it admits a "continuum" interpretation as a renormalizable NC field theory. The physical content of such a fixed point would require to study e.g. 4-point functions and the running of the coupling, which is beyond the scope of this work. Note however that if we assume that g becomes larger with increasing cutoff as in the commutative case, this would mean that the low-energy coupling corresponding to g_c is small, and hence in a physically interesting regime.

These results and in particular the existence of a phase transition are roughly consistent with the results of [6–8], but not precisely; for example [6–8] argue for a first-order phase transition using self-consistent Hartree-Fock approximation resp. a one-loop RG analysis, while we find a higher-order transition using a non-perturbative matrix model result. The approaches are thus quite orthogonal: in the present approach the interaction is treated exactly while the kinetic term is approximated, whereas [6–8] do the opposite. The fact that one always finds a phase-transition is quite encouraging. It is also remarkable that θ does not enter our result (95); recall however that we always assume $\Lambda^2\theta \gg 1$ and (154). In particular, one would expect a standard Ising transition to a uniform symmetry breaking state at small $\theta\Lambda^2$; however our approach is not valid in that regime, and we should therefore not expect to see this transition.

In dimensions higher than 4, (86) together with $\alpha_0^2 = O(\Lambda^{d-2}) = O(N^{d/2-1})$ implies that

[13] we find indeed $m_R = 0$ at g_c, which should however not necessarily be taken at face value

$g = O(N^{-\frac{d}{2}+2})$ must be fine-tuned in order to stay in the weakly-coupled phase. This is consistent with the expected non-renormalizability, and we will not pursue this any further.

6.2 2 dimensions

In 2 dimensions our approach is more delicate as the planar diagrams are only logarithmically divergent. To be safe we should allow only finite m_R while $\Lambda \to \infty$, see Appendix B. Using (22) and (86) for $\alpha_0 = \alpha_0(m_R)$ and $\mathcal{N} = N$, we have

$$m_R^2 = \frac{\Lambda^2}{e^{\pi\alpha_0^2} - 1} = \frac{\Lambda^2}{e^{2\sqrt{\frac{2\Lambda^2}{g}}} - 1}. \tag{99}$$

Plugging this in (88) and using the relation $\Lambda^2 = 2\frac{N}{\theta}$ for the regularization using the fuzzy sphere gives

$$m^2 + \frac{3}{2}\sqrt{\frac{g}{\pi}}\,\Lambda = \frac{\Lambda^2}{e^{2\sqrt{\frac{2\Lambda^2}{g}}} - 1} = m_R^2. \tag{100}$$

This would be consistent with a scaling

$$m^2 \sim \Lambda^2, \quad g \sim \Lambda^2 \tag{101}$$

on this critical line[14]. However, the basic assumptions of Sections 3.1 and Appendix B are no longer valid in this case, since then $m_R^2 \propto \Lambda^2$. Our approach should be most reliable if m_R remains finite, which implies through (99) that $g \sim 2(\frac{\Lambda}{\ln\Lambda})^2$ and therefore (100) $m^2 \sim -\frac{3}{\pi}\frac{\Lambda^2}{\ln\Lambda}$. Since we cannot solve (100) in closed form, let us assume that g is smaller but not much smaller that $g \approx 2(\frac{\Lambda}{\ln\Lambda})^2$, so that we can neglect the term on the rhs of (100). Then the critical line should be given approximately by

$$m^2 = -\frac{3}{2}\sqrt{\frac{g}{\pi}}\,\Lambda, \tag{102}$$

or a slightly modified formula using the fuzzy torus regularization (due to the different propagator). Unfortunately we cannot compare this with the numerical results of [10], who consider a different scaling. However we can compare it to some extent with the numerical results of [9] on the fuzzy sphere, and find reasonable agreement. This will be done in the following section.

If g scales like (101), one can use a simpler argument due to [9] neglecting the kinetic term altogether, and replace the action by the pure potential model $TrV(\phi)$; hence this phase was denoted as "matrix phase" in [9]. Note that indeed the scaling (101) is appropriate for the matrix model (58) without the kinetic term. This amounts to identifying

$$m'^2 = 2\pi\theta\frac{m^2}{N}, \quad g' = 2\pi\theta\frac{g}{N}, \tag{103}$$

which would predict a phase transition at $(2\pi\frac{m^2\theta}{N})^2 = 8\pi\frac{g\theta}{N}$, i.e.

$$m^2 = -\frac{1}{\pi}\sqrt{2g}\,\Lambda. \tag{104}$$

[14]note that both m^2 and g have dimension of $mass^2$; nevertheless, this is a rather strange scaling in 2 dimensions. This phase transition is very far from the weak coupling phase

6.2.1 The fuzzy sphere

For the case of the fuzzy sphere, we consider the action

$$S = \frac{4\pi R^2}{N} Tr(\phi\Delta\phi + r\phi^2 + \lambda\phi^4) \tag{105}$$

using the (redundant) parameters r, λ, R following [9]. The eigenvalues of $\Delta = \frac{\mathcal{L}^2}{R^2}$ are $l(l+1)/R^2 = p^2$, with cutoff $\Lambda = p_{max} = N/R$. Comparing with (58), the above parameters are related to the coupling constants m and g in (58) via

$$\theta = \frac{2R^2}{N}, \qquad 2r = m^2, \qquad 4\lambda = g. \tag{106}$$

It turns out that α_0^2 for the fuzzy sphere (141) agrees precisely with the result for $\mathbb{R}^2_\theta$,

$$\alpha_0^2(m_R) = \frac{1}{\pi}\ln(1 + \frac{\Lambda^2}{m_R^2}). \tag{107}$$

We can therefore apply (102), assuming that $\lambda \ll (\frac{\Lambda}{\ln\Lambda})^2$; in particular we can fix $\lambda = 1$ as in [9]. Then the critical line is

$$\frac{r}{N} = -\frac{3}{2\sqrt{\pi}}\,\frac{1}{R} \;\approx\; -0.846\,\frac{1}{R}. \tag{108}$$

Note that R is a free parameter corresponding to θ, and we should take at least $R = O(\sqrt{N})$.

Let us compare his with Martin's disordered-matrix transition [9]: he finds numerically a phase transition for (see eq. (46) in [9])

$$\frac{r}{N} \approx -0.56\,\frac{1}{R} \tag{109}$$

for large R. This is reasonably close to our analytical result. Recall that we cannot expect our prediction (108) to be exact since the eigenvalue distribution is quite far from Wigner's law, and moreover the arguments in Section 2 are weaker in 2 dimensions compared to 4 dimensions since the crucial divergences are only logarithmic. Furthermore g strongly violates $g \sim 2(\frac{\Lambda}{\ln\Lambda})^2$ and therefore $m_R \to 0$, so that the replacement of the kinetic term with (34) is not justified here. The pure matrix model discussed above (103) would give

$$\frac{r}{N} = -\frac{\sqrt{2}}{\pi}\,\frac{1}{R} \;\approx\; -0.45\,\frac{1}{R}. \tag{110}$$

Hence the numerical results of [9] (taken for $N \approx 30$) appear to be in between the pure potential model and our approach, and our treatment apparently overestimates the kinetic term. This is not too surprising in 2 dimensions in view of the above remarks.

7 Discussion and outlook

We presented a simple non-perturbative approach to scalar field theory on Euclidean non-commutative spaces, based on certain matrix regularizations of $\mathbb{R}^{2n}_\theta$. Starting from a representation of the field $\phi = U(\phi_i)U^{-1}$ in terms of eigenvalues ϕ_i and "angles" U, we observe

that the different behavior of planar and non-planar diagrams due to UV/IR mixing implies a particular eigenvalue density distribution, which can be reproduced by a simple matrix model. This is shown starting with the case of free fields, which can be described by a Gaussian matrix model. Interactions of the form $TrV(\phi)$ can then be included very easily, and modify the eigenvalue distribution. This leads to a picture where the basic properties of the QFT such as correlation length and renormalization are related to its eigenvalue density $\rho(s)$, through a "constrained field theory" with compact configuration space $\mathcal{O}(\phi)$. It also makes the existence of well-behaved scaling limits of NCFT i.e. renormalizability very plausible. We conclude that the eigenvalue sector of non-commutative scalar field theories is goverened by an ordinary matrix model, which provides a simple and intuitive window into the non-perturbative domain.

For weak coupling, this approach provides a new way of computing the renormalization of the potential; in particular, using a very simple approximation we found an expression for the mass renormalization, which coincides with the conventional one-loop calculation. Furthermore, we found a phase transition at strong coupling in the ϕ^4 model in both 2 and 4 dimensions, which is identified with the striped or matrix phase of [6, 9]. This is particularly interesting in the 4-dimensional case since the critical line then terminate at a non-trivial point with $g_c \neq 0$, which is interpreted as an interacting RG fixed-point. The existence of this fixed point can be understood simply by noting that the critical line is characterized by a specific eigenvalue distribution which is different from Wigner's law, which holds for $g = 0$. Therefore the critical line cannot end at $g = 0$. All this suggests that such NC field theories may in fact be more acccesible to analytical tools than their commutative counterparts.

Perhaps the main shortcoming of this approach is the lack of a precise relation between the eigenvalue distribution and the relevant physical parameters, such as correlation length resp. mass and coupling strength. It is quite clear that the leading parameter is the size α of the maximal eigenvalue, which has been used in this paper to extract physical information. To go beyond this approximation may be difficult, and may require e.g. perturbative methods.

There are many other questions and gaps which should be addressed in future work. For example, the assumption that θ_{ij} is non-degenerate (or special) is quite clearly not essential, and a similar approach should also work in odd dimension. Furthermore, a more elaborate analysis of the renormalization in the weakly-coupled regime should be attempted. Another interesting question concerns the relation of this approach with the results of [11] on a modified ϕ^4 model; to address this issue, the above analysis should be repeated with a suitably modified propagator. This will be done elsewhere.

Perhaps the most interesting perspective is the possibility that careful estimates of the contribution of the kinetic term on the orbits $\mathcal{O}(\phi)$ (for eigenvalue distributions different from Wigner's law) should allow to rigorously justify the above picture also in the non-perturbative domain, and in particular the existence of the critical point. One may in particular try to establish renormalizability in this way. This should be facilitated by the fact that $\mathcal{O}(\phi)$ are compact spaces.

Finally, it would of course be extremely interesting to compare all this with numerical results in 4 dimensions, which are not available at this time.

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Appendix A: Regularizations of $\mathbb{R}^{2n}_\theta$

The fuzzy torus T^2_N and $T^2_N \times T^2_N$

A particularly simple regularization of $\mathbb{R}^d_\theta$ which works in any (even) dimensions was proposed in [18], which we review for convenience. Consider a (toroidal) lattice with lattice constant a and N sites in each dimension. We denote its size with

$$L = Na. \tag{111}$$

Since we are on a torus, one should not use the unbounded operators x_j. Instead consider the unitary generators

$$Z_j := e^{\frac{2\pi}{L} i x_j}\,, \qquad Z_i^N = 1. \tag{112}$$

The commutation relations $[x_i, x_j] = i\Theta_{ij}$ then become

$$Z_i Z_j = \exp\left(-\frac{4\pi^2}{L^2} i\Theta_{ij}\right) Z_j Z_i. \tag{113}$$

Rather than going through the most general case, we simply consider

$$\Theta_{ij} = \theta\, Q_{ij} \tag{114}$$

and work out the 4-dimensional case, where

$$Q_{ij} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}. \tag{115}$$

The generalization to any even dimension is obvious. Periodicity implies a quantization of a resp. θ as

$$\frac{Na^2}{\pi} = \theta, \tag{116}$$

assuming that N is odd. The physical momentum is

$$k_i = \frac{2\pi n_i}{aN} \in \left(-\frac{\pi}{a}, \frac{\pi}{a}\right) = (-\Lambda, \Lambda), \tag{117}$$

with UV-cutoff at

$$\Lambda = \frac{\pi}{a} = \sqrt{\frac{\pi N}{\theta}}. \tag{118}$$

This is in qualitative agreement with the scaling (154) obtained using fuzzy $\mathbb{C}P^n$.

In order to write down the action for a scalar field, we also need partial derivatives or shift operators,

$$D_j := e^{a\partial_j}, \qquad D_j Z_i D_j^\dagger = e^{2\pi i \delta_{ij}/N} Z_i\,. \tag{119}$$

In our case, they can be realized as

$$D_1 = (Z_2^\dagger)^{(N+1)/2}, \quad D_2 = (Z_1)^{(N+1)/2} \tag{120}$$

etc. A solution of (113) and $Z_i^N = 1$ in 2 dimensions is given by the unitary "clock and shift" operators (recall that N is odd)

$$Z_1 = \begin{pmatrix} 0 & 1 & & & & \\ & 0 & 1 & & & \\ & & \cdot & \cdot & & \\ & & & \cdot & \cdot & \\ & & & & 0 & 1 \\ 1 & & & & & 0 \end{pmatrix}, \qquad Z_2 = \begin{pmatrix} 1 & & & & \\ & e^{4\pi i/N} & & & \\ & & e^{2(4\pi i/N)} & & \\ & & & \cdot & \\ & & & & \cdot \\ & & & & \end{pmatrix}, \tag{121}$$

which extends to arbitrary even dimensions by taking tensor products. Hence the field ϕ is a hermitean $N^{d/2} \times N^{d/2}$ matrix. The integral is again defined as

$$\int f = (2\pi\theta)^{d/2} Tr(f). \tag{122}$$

One can then define the "plane waves"

$$\phi_{\vec{n}} = \frac{1}{N^{d/4}} \prod_{i=1}^{d} (Z_i)^{n_i} \; (\prod_{j<i} e^{2\pi i Q_{ij} n_i n_j/N}) \tag{123}$$

which satisfy

$$Tr(\phi_{\vec{n}}^\dagger \phi_{\vec{n}'}) = \delta_{\vec{n}\vec{n}'}, \qquad \phi_{\vec{n}}^\dagger = \phi_{-\vec{n}} \tag{124}$$

for $n_i \in [-(N-1)/2, (N+1)/2]$. They form a basis of the "space of functions" $Mat(N^{d/2}, \mathbb{C})$. Using

$$D_i \phi_{\vec{n}} D_i^\dagger = \exp(2\pi i n_i/N)\; \phi_{\vec{n}} \tag{125}$$

one can write down the discretized lattice-version of (1),

$$S[\phi] \;=\; (2\pi\theta)^{d/2}\,\mathrm{Tr}\left[\frac{1}{a^2}\sum_{j=1}^{d}(\phi^2 - D_j\phi D_j^\dagger\phi) + \frac{m^2}{2}\phi^2 + \frac{g}{4}\phi^4\right]. \tag{126}$$

For hermitean $\phi = \sum p_{\vec{n}} \phi_{\vec{n}}$ with $p_{\vec{n}} = p_{-\vec{n}}^\dagger$, the kinetic term becomes

$$\begin{aligned} \frac{1}{a^2} Tr \sum_i \left(D_i \phi D_i^\dagger - \phi\right)^2 &= \frac{2}{a^2}\sum_k |p_k|^2 \sum_j (1 - \cos(k_j a)) \\ &= \sum_k |p_k|^2 (\sum_j k_j^2 + O(a^2 k^4)). \end{aligned} \tag{127}$$

The propagator is therefore

$$\langle p_{\vec{k}} p_{\vec{k}'}^\dagger \rangle = \delta_{\vec{k}\vec{k}'} \frac{1}{\frac{2}{a^2}\sum_j (1-\cos(k_j a)) + m^2}, \tag{128}$$

and the phase factor for the nonplanar diagrams is obtained from

$$\phi_{\vec{k}}\phi_{\vec{k}'} = \exp(-i\theta \sum_{i<j} k_i Q_{ij} k_j')\ \phi_{\vec{k}'}\phi_{\vec{k}}. \tag{129}$$

α_0^2 on the fuzzy torus

Due to the different behavior of the propagator for large momenta, α_0^2 on the fuzzy tori will be somewhat different from regularizations using a sharp cutoff, such as on fuzzy $\mathbb{C}P^n$. In 2 dimensions, we should compute more carefully

$$\begin{aligned} \langle \int d^2x \phi^2(x) \rangle &= V \int_{-\pi/a}^{\pi/a} \frac{d^2p}{(2\pi)^2} \frac{a^2}{2\sum_i (1-cos(p_i a)) + m^2 a^2} \\ &= V \int_0^{\pi} \frac{d^2r}{(2\pi)^d} \frac{1}{\sum_i (1-cos(r_i)) + m^2a^2/2} \end{aligned} \tag{130}$$

and in 4 dimensions

$$\begin{aligned} \langle \int d^4x \phi^2(x) \rangle &= V \int_{-\pi/a}^{\pi/a} \frac{d^4p}{(2\pi)^4} \frac{a^2}{2\sum_i (1-cos(p_i a)) + m^2 a^2} \\ &= \frac{V}{\pi^2} \Lambda^2 \int_0^{\pi} \frac{d^4r}{(2\pi)^4} \frac{1}{\sum_i (1-cos(r_i)) + m^2a^2/2} \end{aligned} \tag{131}$$

for the above regularization. In particular, for $m=0$ one has

$$\alpha_0^2(m) \le \alpha_0^2(m=0) = \frac{4}{\pi^2} \Lambda^2 \int_0^{\pi} \frac{d^4r}{(2\pi)^4} \frac{1}{\sum_i (1-cos(r_i))} = \frac{0.31}{4\pi^2} \Lambda^2 \tag{132}$$

numerically. This is needed e.g. to determine the critical coupling g_c for the ϕ^4 model in 4 dimensions.

The fuzzy sphere

The algebra S_N^2 of functions on the fuzzy sphere [16] is the finite algebra generated by Hermitian operators $x_i = (x_1, x_2, x_3)$ satisfying the defining relations

$$[x_i, x_j] = i\Lambda_N \epsilon_{ijk} x_k, \tag{133}$$

$$x_1^2 + x_2^2 + x_3^2 = R^2 \tag{134}$$

where R is an arbitrary radius. The noncommutativity parameter Λ_N is of dimension length, and is quantized by

$$\frac{R}{\Lambda_N} = \sqrt{\frac{N^2-1}{4}}, \quad N = 1, 2, \cdots \tag{135}$$

This can be easily understood: (133) is simply the Lie algebra $su(2)$, whose irreducible representation have dimension N. The Casimir of the N-dimensional representation is quantized, and related to R^2 by (134) and (135). Thus the fuzzy sphere is characterized by its radius R and the "noncommutativity parameters" N or Λ_N. The algebra of "functions" S^2_N is simply the algebra $Mat(N)$ of $N \times N$ matrices. It is covariant under the adjoint action of $SU(2)$, under which it decomposes into the irreducible representations with dimensions $(1) \oplus (3) \oplus (5) \oplus ... \oplus (2N-1)$. The integral of a function $f \in S^2_N$ over the fuzzy sphere is

$$\int \phi(x) = \frac{4\pi R^2}{N} Tr[\phi(x)], \tag{136}$$

which agrees with the integral on S^2 in the large N limit and is invariant under the $SU(2)$ rotations. The dimensionless coordinates $\lambda_i = x_i/\Lambda_N$ generate the rotation operators J_i:

$$J_i f = [\lambda_i, f]. \tag{137}$$

One can now easily write down actions for scalar fields, such as

$$S = \frac{4\pi R^2}{N} Tr\left(\frac{1}{2}\phi\Delta\phi + \frac{1}{2}m^2\phi^2 + \frac{1}{4}g\phi^4\right) = 2\pi\theta\ Tr\left(\frac{1}{2}\phi\Delta\phi + \frac{1}{2}m^2\phi^2 + \frac{1}{4}g\phi^4\right) \tag{138}$$

were ϕ is a Hermitian matrix, and $\Delta\phi = \frac{1}{R^2} J_i J_i \phi$ is the Laplace operator. The last form is obtained by defining θ through

$$R^2 = \frac{N\theta}{2}. \tag{139}$$

There are 2 obvious $N \to \infty$ limits: 1) the conventional, commutative S^2 limit keeping R fixed, and 2) the limit of the NC plane $\mathbb{R}^2_\theta$, which can be obtained by keeping θ constant: then the tangential coordinates $x_{1,2}$ satisfy at the north pole the commutation relations

$$[x_1, x_2] = i\frac{2R}{N} x_3 = i\frac{2R}{N}\sqrt{R^2 - x_1^2 - x_2^2} \approx i\theta. \tag{140}$$

Note that θ^{-1} now determines the basic (NC) scale of the NC field theory, replacing the radius R.

If one considers non-planar loops, the oscillatory behavior due to the $6j$ symbols sets in for angular momenta $l^2 \approx N$ and was studied in [24]. Note that this corresponds to $p^2 = O(\theta^{-1})$ in the NC case, but to $p \to \infty$ in the commutative case. Therefore the non-oscillatory domain is divergent in the commutative limit and prevents a matrix behavior. This "low-energy" sector is suppressed in the NC case if θ is kept finite.

α_0^2 for the fuzzy sphere

The arguments of Section 3.1 for the eigenvalue distribution go through here as well provided the oscillating behavior of the non-planar diagrams is strong enough. Even though there are nontrivial effects even for finite R resp. $\theta \propto 1/N$ [24], they are sufficiently strong

for our purpose only if θ is kept finite. To find the appropriate $\alpha_0(m)$, consider for $g = 0$

$$\begin{aligned}
\langle\int_{S^2}\phi^2\rangle &= \sum_{l,m}\int_{S^2}\frac{Y^{lm}\cdot Y^{l-m}}{l(l+1)/R^2+m^2} = \sum_{l=1}^{N}\frac{2l+1}{l(l+1)/R^2+m^2} \\
&= R\sum_{l=1}^{N}\frac{(2l+1)/R}{l(l+1)/R^2+m^2} = 2R^2\int_0^{N/R}dx\frac{x}{x^2+m^2} \\
&= R^2\ln(1+\frac{N^2}{m^2R^2}) = \frac{V}{4\pi}\ln(1+\frac{\Lambda^2}{m^2})
\end{aligned} \tag{141}$$

in the large N limit, using $V = 4\pi R^2$ and $x = \frac{l+1/2}{R}$ and $\Lambda = \frac{N}{R}$. This agrees precisely with (11), so that the same $\alpha_0(m)$ as in (22) can be used.

Fuzzy $\mathbb{C}P^n$

The construction of fuzzy $\mathbb{C}P^n$ [27–30] is analogous to the case of fuzzy $S^2 \cong \mathbb{C}P^1$. Since $\mathbb{C}P^n$ is an adjoint orbit of $SU(n+1)$, it is a compact symplectic space and can be quantized in terms of finite matrix algebras $Hom(V_N)$ where V_N are suitable representations of $su(n+1)$. To identify the correct representations V_N of $su(n+1)$, we must match the space of harmonics on classical $\mathbb{C}P^n$

$$\mathcal{C}^\infty(\mathbb{C}P^n) = \mathop{\oplus}_{p=0}^{\infty} V_{(p,0,\dots,0,p)}. \tag{142}$$

with the decomposition of $Hom(V_N)$. It is easy to show that indeed

$$Hom(V_N) = V_N \otimes V_N^* \cong \mathop{\oplus}_{p=0}^{N} V_{(p,0,\dots.,0,p)} \tag{143}$$

for

$$V_N := V_{(N,0,\dots,0)}.$$

Here $V_{(l_1,\dots,l_n)}$ denotes the highest weight irrep of $su(n+1)$ with highest weight $l_1\Lambda_1+...+l_n\Lambda_n$ where Λ_k are the fundamental weights. One can therefore define the algebra of functions on the fuzzy projective space by

$$\mathbb{C}P^n_N := Hom_{\mathbb{C}}(V_N) = Mat(\mathcal{N},\mathbb{C}) \tag{144}$$

with

$$\mathcal{N} = \frac{(N+n)!}{N!n!} \approx \frac{N^n}{n!}. \tag{145}$$

The functions on fuzzy $\mathbb{C}P^n$ have a UV cutoff given by N. Scalar fields on $\mathbb{C}P^n_N$ are elements in $Hom_{\mathbb{C}}(V_N)$, and the integral is given by the suitably normalized trace over V_N. The coordinate functions x_a for $a = 1,...,n^2+2n$ on fuzzy $\mathbb{C}P^n$ are given by suitably rescaled generators of $su(n+1)$ acting on V_N. One finds [30]

$$[x_a, x_b] = i\Lambda_N f_{abc}\, x^c, \qquad g^{ab}x_ax_b = R^2, \tag{146}$$

$$d_c^{ab}x_ax_b = (n-1)(\frac{N}{n+1}+\frac{1}{2})\Lambda_N\, x_c. \tag{147}$$

for

$$\Lambda_N = \frac{R}{\sqrt{\frac{n}{2(n+1)}N^2 + \frac{n}{2}N}}. \tag{148}$$

For large N, this reduces to the defining relations of $\mathbb{C}P^n \subset \mathbb{R}^{n^2+2n}$. On the other hand, scaling the radius as

$$R^2 = N\theta \frac{n}{n+1} \tag{149}$$

near a given point (the "north pole") of $\mathbb{C}P^2_N$ gives $\mathbb{R}^{2n}_\theta$ with $U(n)$ invariant θ_{ij}, similarly as for the fuzzy sphere. We refer to [29] for further details.

The Laplacian on fuzzy $\mathbb{C}P^n$ is proportional to the quadratic Casimir of $su(n+1)$ acting on the functions,

$$\Delta(\phi) = \frac{c}{R^2}\, J_a J_a \phi, \tag{150}$$

where J_a generates the $SU(n+1)$ rotations and $c = \frac{2n}{n+1}$. It has eigenvalues

$$\Delta f_k(x) = c\frac{k(k+n)}{R^2}\, f_k(x) \tag{151}$$

for $f_k(x) \in V_{(k,0,\ldots,0,k)}$ according to the decomposition (143). The multiplicity for given k is

$$dim(V_{(k,0,\ldots,0,k)}) = \frac{(k+n-1)!^2}{k!^2(n-1)!^2}\frac{2k+n}{n} \approx \frac{2}{(n-1)!^2 n}\, k^{2n-1} \tag{152}$$

for $k \gg n$. To find the appropriate $\alpha_0(m)$, consider

$$\begin{aligned}
\langle \int_{\mathbb{C}P^n} \phi^2 \rangle &= \sum_{k,m} \int_{\mathbb{C}P^n} \frac{Y^{km}\cdot Y^{k-m}}{ck(k+n)/R^2 + m^2} = \frac{2}{(n-1)!^2 n} \sum_{k=1}^{N} \frac{k^{2n-1}}{ck(k+n)/R^2+m^2} \\
&= \frac{2(R/\sqrt{c})^{2n-1}}{(n-1)!^2 n} \sum_{k=1}^{N} \frac{(k\sqrt{c}/R)^{2n-1}}{ck(k+n)/R^2+m^2} = \frac{2(R/\sqrt{c})^{2n}}{(n-1)!n!} \int_0^\Lambda dx \frac{x^{2n-1}}{x^2+m^2} \\
&= \frac{V}{2^{2n-1}\pi^n (n-1)!} \int_0^\Lambda dx \frac{x^{2n-1}}{x^2+m^2}
\end{aligned} \tag{153}$$

in the large N limit, where we denote the basis again with Y^{km} and used $V = Vol(\mathbb{C}P^n) = \left(\frac{2(n+1)}{n}\right)^n \frac{\pi^n}{n!}\, R^{2n}$ and $x = \sqrt{c}\frac{k}{R}$ and

$$\Lambda = \sqrt{c}\frac{N}{R} = \sqrt{\frac{2n}{n+1}}\,\frac{N}{R} = \sqrt{\frac{2N}{\Theta}}. \tag{154}$$

This agrees with precisely with (11) and generalizes the results for the fuzzy sphere. Furthermore, putting (154) and (149) together gives

$$\int_{\mathbb{C}P^n_N} := \frac{V}{\mathcal{N}} Tr(.) \to (2\pi\theta)^n Tr(.) \tag{155}$$

in the above scaling limit. Therefore this rescaled fuzzy $\mathbb{C}P^n$ is a perfect regularization of $\mathbb{R}^{2n}_\theta$. It has a sharp mumentum cutoff at Λ, and the same $\alpha_0(m)$ as in (22) can be used.

Appendix B: Justification of (16)

Consider first $d \geq 3$. Then

$$R_{NP} := \frac{\langle \frac{1}{V} \int d^d x \phi(x)^{2n} \rangle_{Non-Planar}}{\langle \frac{1}{V} \int d^d x \phi(x)^2 \rangle^n} \approx \frac{1}{c} \sum \int_0^1 \frac{d^d k_1'}{(k_1')^2} \dots \frac{d^d k_n'}{(k_n')^2} e^{i\Lambda^2 \sum k_i' \Theta k_j'}, \tag{156}$$

and we assume that $\Lambda^2 \gg m_\theta^2$, $\Lambda^2 \gg m^2$. Here the integration domains have been rescaled to be unit balls in momentum space for all diagrams in the numerator and in the denominator; then the denominator can be estimated by the planar contribution which gives a finite contribution c (after the rescaling), which will be omitted. To proceed, let $\vec{v} \propto (k_1 ...., k_{nd}) \in \mathbb{R}^{nd}$ denote a unit vector in $\mathbb{R}^{nd}$ with norm $\|\vec{v}\| = 1$, and consider generalized spherical coordinates such that $d^d k_i ... d^d k_n = d\Omega(\vec{v})\, r^{nd-1} dr$ in $\mathbb{R}^{nd}$. Then

$$\int_0^1 \frac{d^d k_1'}{(k_1')^2} \dots \frac{d^d k_n'}{(k_n')^2} e^{i\Lambda^2 \sum k_i' \Theta k_j'} = \int d\Omega(\vec{v}) \int_0^{O(1)} dr\ r^{nd-1} \frac{e^{i\Lambda^2 r^2 \vec{v}\Theta\vec{v}}}{(k_1')^2 \dots (k_n')^2} \tag{157}$$

in simplified notation. The important point is that the radial integral $\int dr$ is oscillatory, with increasing frequency. For $d \geq 3$, the radial integral behaves as

$$\int_0^{O(1)} dr\ r^{n(d-2)-1} e^{i\Lambda^2 r^2 \vec{v}\Theta\vec{v}} = \int_0^{O(1)} du\ u^{n(d-2)/2-1} e^{i\Lambda^2 u \vec{v}\Theta\vec{v}} \tag{158}$$

where $r^2 = u$. Clearly the (alternating) contributions increase with u. The integral is therefore estimated by the "last oscillation" which is of order $min\{1, O(1/(\Lambda^2 \vec{v}\Theta\vec{v}))\}$. We can exclude the region $\{\vec{v}\Theta\vec{v} < 1/\Lambda\}$, whose volume goes like $O(1/\Lambda)$ as $\Lambda \to \infty$ (ignoring $\log \Lambda$-corrections). Therefore

$$R_{NP} \leq O(1/\Lambda) \tag{159}$$

ignoring possible log-corrections[15], which establishes our claim.

For $d = 2$, we apply the same analysis to the numerator of (156). The radial integral has again the form (158), but it is now dominated by small u, i.e. the "first half-oscillation". We therefore have to reintroduce the masses which provide an IR cutoff. To estimate this, we go back to the original form

$$\begin{aligned} \langle \frac{1}{V} \int d^d x \phi(x)^{2n} \rangle_{Non-Planar} &= \int_0^\Lambda \frac{d^2 k_1}{(k_1)^2 + m^2} \dots \frac{d^2 k_n}{(k_n)^2 + m^2} e^{i \sum k_i \Theta k_j} \\ &\approx \int_{\sum k_i \Theta k_j < \pi} \frac{d^2 k_1}{(k_1)^2 + m^2} \dots \frac{d^2 k_n}{(k_n)^2 + m^2} \\ &\approx f(\ln(\frac{m}{m_\theta})). \end{aligned} \tag{160}$$

Indeed the result can only depend on the ratio $\frac{m}{m_\theta}$ where

$$m_\theta^2 = \frac{1}{\theta} \tag{161}$$

[15] this is probably not the best possible estimate

is the NC mass scale, and in 2 dimensions it will depend only logarithmically on this ratio (for example, $\langle\frac{1}{V}\int d^dx\phi(x)^4\rangle_{Non-Planar} = O(\ln(\frac{m}{m_\theta})^2)$. On the other hand,

$$\langle\frac{1}{V}\int d^2x\phi(x)^{2n}\rangle_{Planar} = O(\ln(\frac{\Lambda}{m})^n), \tag{162}$$

Hence for fixed $\frac{m}{m_\theta}$ and $\Lambda \to \infty$, the eigenvalue distribution is again given by the Gaussian matrix model (34), however we expect the fluctuations to be larger in this case than for $d = 4$.

It is interesting to consider also the commutative scaling limit of the fuzzy sphere, in view of the "non-commutative anomaly" found in [24]. In that case, we can set $\theta = 1/NR^2$ and therefore $m_\theta^2 = O(N) = O(\Lambda)$. Then both planar and non-planar contributions have the same logarithmic behavior, and there is no well-defined eigenvalue distribution in that case. This was to be expected since we considered the commutative limit; however the "non-commutative anomaly" indicates some NC behavior in that case too, which is apparently too weak to induce a distinct eigenvalue distribution. The situation is different in 4 dimensions, and a well-defined eigenvalue distribution may well exist in the commutative scaling limit.

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